

MODULI SPACES OF SEMITORIC SYSTEMS

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ABSTRACT. Recently Pelayo-Vũ Ngọc classified semitoric integrable systems in terms of five symplectic invariants. Using this classification we define a family of metrics on the space of semitoric integrable systems. The family is parameterized by two choices. The induced topology does not depend on these choices. The resulting metric space is incomplete and we construct the completion.

1. INTRODUCTION AND OVERVIEW OF RESULTS

The results of Atiyah, Guillemin-Sternberg, and Delzant in the 1980s completely classify compact toric integrable systems. Recently Pelayo-Pires-Ratiu-Sabatini [27] used this classification to define a natural metric on the moduli space of such systems. The goal of this paper is to extend their construction to semitoric systems, that is to construct a metric on the moduli space of semitoric integrable systems and study its properties. A *toric integrable system* is a connected compact¹ $2n$ -dimensional symplectic manifold (M, ω) with a momentum map

$$F = (f_1, \dots, f_n) : M \rightarrow \mathbb{R}^n$$

such that all of the flows generated by f_i , $i = 1, \dots, n$, are periodic of a fixed period. Atiyah [1] and Guillemin-Sternberg [11] proved that the image $F(M)$ is a convex polytope and later Delzant [5] showed that the polytope also satisfies certain further conditions making it a *Delzant polytope* (see Definition 1.8) and that the polytope determines the isomorphism class of (M, ω) . Delzant also showed that given any Delzant polytope one can construct a toric integrable system having that polytope as the image of its momentum map. In short, there exists a bijective correspondence between the set of Delzant polytopes and the set of toric integrable systems modulo isomorphism.

Thus, questions about toric integrable systems may be answered by instead examining the space of Delzant polytopes. In [27] the authors define a metric on the space of Delzant polytopes via the volume of the symmetric difference² and pull this back to produce a metric on the moduli space of toric integrable systems. In particular, the metric provides this space with a natural topology and in [9] Figalli and Pelayo use this topology to explore the continuity properties of the maximal toric ball packing density function³. Maximal ball packings have been of great interest in symplectic topology for many years [2, 3, 17] and there is particular interest in the toric case [21, 28].

¹For this paper we will define toric integrable systems to be compact because this is the definition used in [27]. Delzant's original classification required the toric manifold to be compact, although recently his results have been extended to the non-compact setting [16].

²The construction of this metric is related to the Duistermaat-Heckman measure [6].

³see Section 4 for a discussion of this question in the semitoric case.

In [22, 23] Pelayo and Vũ Ngọc provide a complete classification for a broader class of integrable systems, those known as semitoric, in terms of a collection of several invariants.

Definition 1.1. A *semitoric integrable system* is a 4-dimensional connected symplectic manifold (M, ω) with a momentum map $F = (J, H) : M \rightarrow \mathbb{R}^2$ satisfying

- (1) the function J is a proper momentum map for a Hamiltonian circle action on M ;
- (2) F has only non-degenerate singularities (in the sense of Williamson[31]) without real-hyperbolic blocks.

Notice that Definition 1.1 does require that M be 4-dimensional but there is much more freedom in the choice of momentum map (compared to toric systems) and M is not required to be compact. Condition (2) means that if $p \in M$ is a critical point of F then there exists some 2×2 matrix B such that $\tilde{F} = B \circ (F - F(p))$ is given by one of three different standard forms which correspond to the classification of the singularity at p (due to Eliasson [7, 8]). There must exist a local symplectic chart (x, y, η, ξ) centered at p which puts \tilde{F} into one of the three possible singularity types:

- (1) transversally elliptic singularity: $\tilde{F}(x, y, \eta, \xi) = (\eta + \mathcal{O}(\eta^2), \frac{x^2 + \xi^2}{2}) + \mathcal{O}((x, \xi)^3)$;
- (2) elliptic-elliptic singularity: $\tilde{F}(x, y, \eta, \xi) = (\frac{x^2 + \xi^2}{2}, \frac{y^2 + \eta^2}{2}) + \mathcal{O}((x, \xi, y, \eta)^3)$;
- (3) focus-focus singularity: $\tilde{F}(x, y, \eta, \xi) = (x\xi + y\eta, x\eta - y\xi) + \mathcal{O}((x, \xi, y, \eta)^3)$.

The focus-focus singular points will be the ones we are most interested in when reviewing the invariants and when defining the metric.

Definition 1.2. A semitoric integrable system $(M, \omega, F = (J, H))$ is said to be *simple* if the following generic⁴ assumption holds: if $p \in M$ is a focus-focus critical point of F then it is the only focus-focus critical point in the set $J^{-1}(J(p))$.

Any semitoric system has only finitely many focus-focus critical points (see Section 1.2.1) so we will denote them by $c_1, \dots, c_{m_f} \in M$. In a simple system we have that the values $J(c_1), \dots, J(c_{m_f})$ are pairwise distinct and thus we will assume throughout this article that they are ordered so $J(c_1) < \dots < J(c_{m_f})$. All semitoric systems studied in this article are assumed to be simple.

Definition 1.3. Suppose that $(M_1, \omega_1, F_1 = (J_1, H_1))$ and $(M_2, \omega_2, F_2 = (J_2, H_2))$ are semitoric systems. An *isomorphism of semitoric systems* is a symplectomorphism $\phi : M_1 \rightarrow M_2$ such that $\phi^*(J_2, H_2) = (J_1, f(J_1, H_1))$ where $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a smooth function such that $\frac{\partial f}{\partial H_1}$ nowhere vanishes. We denote by \mathcal{T} the space of simple semitoric systems modulo isomorphism.

Our goal is to define a metric on the space of invariants and thus induce a metric on \mathcal{T} . In order to do this we must first understand the invariants and the classification theorem of Pelayo-Vũ Ngọc.

1.1. Structure of paper. First, in Section 1.2 we describe the invariants of a semitoric integrable system and state the Pelayo-Vũ Ngọc classification theorem and in Section 1.3 we are able to describe the metric and state the main result of this paper, Theorem A. The content of this Theorem is split into Proposition 2.13 and Proposition 3.15 which are the subject of each of Sections 2 and 3. Specifically, in Section 2 we prove the function in

⁴A similar (but weaker) assumption is generic according to Zung [32], that each fiber $F^{-1}(c)$ for $c \in \mathbb{R}^2$ contains at most one critical point $p \in M$.

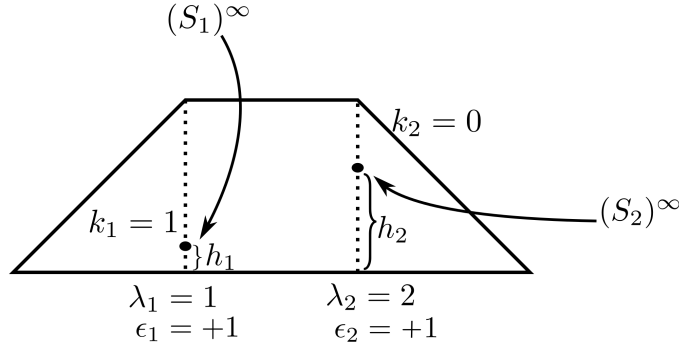


FIGURE 1. The complete invariant is a collection of polygons with distinguished points $\{c_1, \dots, c_j\}$ each labeled with extra information: a Taylor series $(S_j)^\infty$, an integer k_j known as the twisting index (only defined up to addition by a common integer as is described in Section 1.2.4), and an element $\epsilon \in \{-1, +1\}$ known as the cut direction. There is one polygon in the family for each possible choice of cut directions and each allowed choice of twisting indices.

question is a metric and we also describe a metric on general Taylor series (Section 2.1). After this in Section 3 we construct the completion of the metric space and in Section 4 we propose several possible applications of a metric and in particular a topology on the space of semitoric systems and state several related open problems.

1.2. The classification of semitoric integrable systems. In the compact toric case the integrable systems could be classified in terms of Delzant polytopes. In the semitoric case a polygon plays a role but the complete invariant must contain more information. Loosely speaking, the complete invariant is a collection of convex polygons in \mathbb{R}^2 (which may not be compact) with a finite number of distinguished points corresponding to the focus-focus singularities each labeled by a Taylor series and an integer⁵ (See Figure 1).

1.2.1. The number of singular points invariant. In [19, Theorem 1] Vũ Ngọc proves that a semitoric system has finitely many focus-focus singular points. Thus to a system we may associate an integer $0 \leq m_f < \infty$ which is the total number of focus-focus points in the system. Clearly the singular points are preserved by isomorphism so this is an invariant of the system.

Definition 1.4. For any nonnegative integer $m_f \in \mathbb{Z}_{\geq 0}$ let \mathcal{T}_{m_f} denote the collection of simple semitoric systems with m_f focus-focus points modulo semitoric isomorphism.

Toric systems, which have no focus-focus singular points, correspond to a subset of \mathcal{T}_0 ⁶. There is some subtlety in this correspondence because of the difference between toric and semitoric isomorphisms, see Section 2.5, but it can be seen that the topology on toric systems

⁵The integer labeling each point is the twisting index, and these labels are only defined up to addition by a common integer, see Section 1.2.4.

⁶This is a strict subset because, for example, semitoric systems with no focus-focus singular points may be non-compact.

from [27] is compatible with the topology defined in the present article (see Section 2.5 and in particular Corollary 2.15). Also, the elements of \mathcal{T}_1 are known as a Jaynes-Cummings type systems which is studied for instance in [10]. An important example of a Jaynes-Cummings type system is the coupled spin-oscillator which is defined in [4, 14] and studied in detail in [26, 19].

1.2.2. The Taylor series invariant. The next invariant we will study completely classifies the semi-global⁷ structure of a focus-focus critical point up to isomorphism [18]. It is defined in terms of the length of certain flow lines of the Hamiltonian vector fields for the components of the momentum map. The details can be found in [18, 24].

Definition 1.5. Let $\mathbb{R}[[X, Y]]$ refer to the algebra of real formal power series in two variables and let $\mathbb{R}[[X, Y]]_0 \subset \mathbb{R}[[X, Y]]$ be the subspace of series $\sum_{i,j \geq 0} \sigma_{i,j} X^i Y^j$ which have $\sigma_{0,0} = 0$ and $\sigma_{0,1} \in [0, 2\pi)$.

The Taylor series invariant is one element of $\mathbb{R}[[X, Y]]_0$ for each of the m_f focus-focus points.

1.2.3. The affine invariant and the twisting index invariant. In this section we define the affine invariant of semitoric systems and also the twisting index. The affine invariant is similar to the polygon from Delzant's result, except in this case we instead have a family of polygons related by specific linear transformations. The twisting index describes how each critical point sits with respect to a privileged momentum map. These two invariants will be described together because the twisting indices which label each critical point will be defined only up to the addition of a common integer related to the choice of polygon. We start with several definitions.

Definition 1.6. A *convex polygonal set*, which for simplicity we will refer to as a *polygon*, is the intersection in \mathbb{R}^2 of (finitely or infinitely many) closed half planes such that on each compact subset of the intersection there are at most finitely many corner points. A convex polygon is rational if each edge is along a vector with rational coefficients. We denote the set of all rational convex polygons by $\text{Polyg}(\mathbb{R}^2)$.

Toric integrable systems are classified by their associated polytopes, but in order to classify semitoric systems we need more than just a polygon. For $\lambda \in \mathbb{R}$ let $\ell_\lambda = \{(x, y) \in \mathbb{R}^2 \mid x = \lambda\}$ and let $\text{Vert}(\mathbb{R}^2)$ be the collection of all vertical lines in \mathbb{R}^2 .

Definition 1.7. A *labeled weighted polygon (of complexity $m_f \in \mathbb{Z}_{\geq 0}$)* is defined to be an element

$$(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f}) \in \text{Polyg}(\mathbb{R}^2) \times (\text{Vert}(\mathbb{R}^2) \times \{-1, +1\} \times \mathbb{Z})^{m_f}$$

with

$$\min_{s \in \Delta} \pi_1(s) < \lambda_1 < \lambda_2 < \dots < \lambda_{m_f} < \max_{s \in \Delta} \pi_1(s)$$

where $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ is projection onto the x -coordinate. We denote the space of labeled weighted polygons of complexity m_f by $\mathcal{LWPoly}_{m_f}(\mathbb{R}^2)$.

Since m_f is the number of singular points of the system, we see that there is a triple $(\ell_{\lambda_j}, \epsilon_j, k_j)$ associated with the singular point c_j for each $j = 1, \dots, m_f$. The polygon invariant (or affine invariant) of a semitoric system with $m_f > 0$ focus-focus points will be an

⁷i.e. in the neighborhood of a fiber $F^{-1}(c) \subset M$ where c is a critical point.

equivalence class of labeled weighted polygons and the twisting invariant refers to the integer label on each point.

Here we will briefly review how the affine invariant is produced. Consider the set $F(M) \subset \mathbb{R}^2$ and recall that in the toric case this would be a Delzant polygon. Let $c_1, \dots, c_{m_f} \in F(M)$ denote the images of the focus-focus points and let $B_r = \text{Int}(F(M)) \setminus \{c_1, \dots, c_{m_f}\}$ which turns out to be precisely the regular values of F ⁸. Let $\lambda_j = \pi_1(c_j)$ for $j = 1, \dots, m_f$ so that ℓ_{λ_j} is a vertical line which passes through c_j . For each $j = 1, \dots, m_f$ cut B_r along the line $\ell_{\lambda_j}^{\epsilon_j}$ which starts at c_j and goes upwards if $\epsilon_j = 1$ and downwards if $\epsilon_j = -1$ to form the set $B_r^{\epsilon_1, \dots, \epsilon_{m_f}}$. Now we have a simply connected set of regular values of F so in fact we can define a global toric momentum map

$$F_{\text{toric}} : F^{-1}(B_r^{\epsilon_1, \dots, \epsilon_{m_f}}) \rightarrow \mathbb{R}^2$$

and $\Delta = \overline{F_{\text{toric}}(B_r^{\epsilon_1, \dots, \epsilon_{m_f}})}$. So the choice of $(\epsilon_j)_{j=1}^{\infty}$ produces different polygons, and in fact there is some freedom in the choice of toric momentum map on $B_r^{\epsilon_1, \dots, \epsilon_{m_f}}$ as well⁹.

Now the distinguished points in each polygon are precisely the image of the focus-focus singular points under F_{toric} . Of course, we are omitting many details in this explanation. Again, the interested reader should see [22, 23].

For $k \in \mathbb{Z}$ let T^k be given by

$$(1) \quad T^k = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \in \text{GL}(2, \mathbb{Z}).$$

Definition 1.8. Let $\Delta \in \text{Polyg}(\mathbb{R}^2)$ be a rational convex polygon. We say that a *vertex* of Δ is a point in the boundary $\partial\Delta$ where the meeting edges are not co-linear. A point is said to be in the *top-boundary* of Δ if it is the top end of a vertical segment formed by intersecting Δ with a vertical line. Suppose that z is a vertex of Δ and (u, v) are a pair of primitive integral vectors starting at z and extending along the direction of the edges which meet at z in the order which makes them oriented. Then the point z is called

- (1) a *Delzant corner* when $\det(u, v) = 1$;
- (2) a *hidden Delzant corner* when it belongs to the top boundary and $\det(u, T^1 v) = 1$;
- (3) a *fake corner* when it belongs to the top boundary and $\det(u, T^1 v) = 0$.

A rational convex polygon $\Delta \in \text{Polyg}(\mathbb{R}^2)$ is called *Delzant* if it is compact and every corner is a Delzant corner. This is the polygon which is relevant for the toric case which we are presently generalizing to the semitoric case.

Now we are nearly ready to define the affine invariant, but to make sure the invariant is unique we must first define the appropriate group action.

⁸According to [22, Remark 3.2] the boundary of B consists of precisely the elliptic points. In fact, B can be viewed as a manifold with corners where the corners are the elliptic-elliptic points and the remainder of the boundary is transversally elliptic points. So the only singular points which can be in the interior of B are the focus-focus points.

⁹Changing the choice of ϵ_j will correspond to the action of the group G_{m_f} and changing the choice of F_{toric} will correspond to the action of \mathcal{G} (see Section 1.2.4).

1.2.4. *The action of $G_{m_f} \times \mathcal{G}$.* It is important that isomorphic systems produce the same invariants, and that choices made when defining the invariants cannot affect which invariant is produced¹⁰. With this in mind we must consider the collection of invariants we have so far modulo a group action.

Notation 1.9. Throughout this article when referring to an m_f -tuple such as $(k_j)_{j=1}^{m_f}$ or $(\epsilon_j)_{j=1}^{m_f}$ for simplicity we will sometimes use vector notation. That is, we may refer to these m_f -tuples as \vec{k} and $\vec{\epsilon}$, respectively. These vectors will always have length m_f .

Let $G_{m_f} = \{-1, +1\}^{m_f}$ and $\mathcal{G} = \{T^k \mid k \in \mathbb{Z}\}$ where T^k is defined as it is above in Equation (1). Suppose that ℓ is a vertical line in \mathbb{R}^2 . Then fix an origin in ℓ which splits \mathbb{R}^2 into two half-spaces and define $t_\ell^k : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ to be the identity on the space left of ℓ and T^k on the right half space (relative to the new origin on ℓ). Now, for $\vec{u} \in \mathbb{Z}^{m_f}$ let $t_{\vec{u}}$ be given by $t_{\vec{u}} = t_{\ell_1}^{u_1} \circ \dots \circ t_{\ell_{m_f}}^{u_{m_f}}$. We define the action of $G_{m_f} \times \mathcal{G}$ on $\mathcal{LWPolyg}_{m_f}(\mathbb{R}^2)$ by

$$(2) \quad ((\epsilon'_j)_{j=1}^{m_f}, T^k) \cdot (\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f}) = (t_{\vec{u}}(T^k \Delta), (\ell_{\lambda_j}, \epsilon'_j \epsilon_j, k + k_j)_{j=0}^{m_f})$$

where $\vec{u} = (\epsilon_j - \epsilon'_j/2)_{j=1}^{m_f}$.

Remark 1.10. Notice that if $(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f})$ is changed (via the action of G_{m_f}) to have $\epsilon'_j \in \{-1, 1\}$ instead of ϵ_j for each $j = 1, \dots, m_f$ then the new polygon is $t_{\vec{u}}(\Delta)$ where $\vec{u} = (\epsilon_j - \epsilon'_j/2)_{j=1}^{m_f} \in \{-1, 0, 1\}^{m_f}$. In fact, this means that the orbit of Δ under the action of G_{m_f} may be written as $(t_{\vec{u}}(\Delta))_{\vec{u} \in \{0,1\}^{m_f}}$ if Δ is the polygon with $\epsilon_j = +1$ for all $j = 1, \dots, m_f$. \oslash

The orbit under this action is the appropriate invariant. It is now clear how the choice of cut direction and constant by which to shift the twisting indices parameterize the collection of all polygons in a given orbit. Notice that the action of $t_{\vec{u}}$ does not necessarily preserve convexity, but it will in the case of the polygons we are interested in (Proposition 1.13).

Definition 1.11. A *labeled Delzant semitoric polygon* is the equivalence class

$$[\Delta_w] \in \mathcal{LWPolyg}_{m_f}(\mathbb{R}^2)/(G_{m_f} \times \mathcal{G})$$

of an element $\Delta_w = (\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})$ ¹¹ satisfying the following.

- (1) The intersection of Δ and any vertical line is either compact or empty;
- (2) each ℓ_{λ_j} intersects the top boundary of Δ ;
- (3) each point in the top boundary which is also in some ℓ_{λ_j} is either a hidden or a fake corner;
- (4) all other corners are Delzant corners.

The space of labeled Delzant semitoric polygons is denoted by

$$\mathcal{DPolyg}_{m_f}(\mathbb{R}^2) \subset \mathcal{LWPolyg}_{m_f}(\mathbb{R}^2)/(G_{m_f} \times \mathcal{G}).$$

¹⁰These choices are explained briefly in Section 1.2.3.

¹¹Notice that we have picked out a preferred representative of each equivalence class by fixing $\epsilon_j = +1$ for each $j = 1, \dots, m_f$. This is a useful strategy we will use many times in this article.

Remark 1.12. The twisting index is only defined up to the addition of a common integer, so the same singular point will have different twisting indices for different elements of a $G_{m_f} \times \mathcal{G}$ -orbit but the relative twisting index between two points is the same for every element of the orbit. \oslash

Any set satisfying Condition (1) is said to have *everywhere finite height*. The following Proposition is just a restatement of [23, Lemma 4.2]. Since a preferred representative Δ can be chosen with $\vec{\epsilon} = (+1, \dots, +1)$ we see that it says that the orbit of Δ under G_{m_f} is a subset of $\text{Polyg}(\mathbb{R}^2)$.

Proposition 1.13. Suppose $\Delta_w \in \mathcal{LWPolyg}_{m_f}(\mathbb{R}^2)$ satisfies items (1)-(4) in Definition 1.11 and $\Delta_w = [(\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})]$. Then for each $\vec{u} \in \{0, 1\}^{m_f}$ the set $t_{\vec{u}}(\Delta_w)$ is convex.

1.2.5. *The volume invariant.* Since the complete invariant includes a polygon with a collection of distinguished points¹² on the lines $\{x = \lambda_j\}$ we must also somehow encode the y -coordinates of these points. By examining the action of $G_{m_f} \times \mathcal{G}$ we can see that the vertical position of the focus-focus points may change, but their height with respect to the bottom of the polygon is constant. Recall that each polygon Δ is the closure of the image of a toric momentum map F (Section 1.2.3).

Definition 1.14. Suppose $[\Delta_w] \in \mathcal{DPolyg}_{m_f}(\mathbb{R}^2)$ with associated toric momentum map F . For $j = 1, \dots, m_f$ we define $0 < h_j < \text{length}(\Delta_w \cap \ell_{\lambda_j})$ by

$$h_j = F(m_j) - \min_{s \in \Delta \cap \ell_{\lambda_j}} \pi_2(s)$$

where $\pi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is projection onto the second coordinate and $(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f}) \in [\Delta_w]$ is any representative.

This is well defined for any choice of polygon in the same equivalence class by [22, Lemma 5.1]. To understand the meaning of the word “volume” in this context see [22].

1.2.6. *The classification theorem.* Now that we have defined all of the invariants we can state the result of Pelayo and Vũ Ngọc found in [22, 23].

Definition 1.15. We define a *semitoric list of ingredients* to be

- (1) a nonnegative integer m_f ;
- (2) a labeled Delzant semitoric polygon $[\Delta_w] = [(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f})]$ of complexity m_f ;
- (3) a collection of m_f real numbers $h_1, \dots, h_{m_f} \in \mathbb{R}$ such that $0 < h_j < \text{length}(\pi_2(\Delta \cap \ell_{\lambda_j}))$ for each $j = 1, \dots, m_f$; and
- (4) a collection of m_f Taylor series $(S_1)^\infty, \dots, (S_{m_f})^\infty \in \mathbb{R}[[X, Y]]_0$.

In other words, a semitoric list of ingredients is a nonnegative integer m_f and an element of $\mathcal{DPolyg}_{m_f}(\mathbb{R}^2) \times \mathbb{R}^{m_f} \times \mathbb{R}[[X, Y]]_0^{m_f}$ where j^{th} element of \mathbb{R} must be in the interval $(0, \text{length}(\pi_2(\Delta \cap \ell_{\lambda_j})))$. Let \mathcal{M} denote the collection of all semitoric lists of ingredients and let \mathcal{M}_{m_f} be lists of ingredients with Ingredient (1) equal to the nonnegative integer m_f .

¹²which are the images under μ of the focus-focus points, see Section 1.2.3.

Notice how the ingredients interact in Definition 1.15. Ingredient (1) determines the number of copies of each other ingredient and Ingredient (3) is in an interval determined by Ingredient (2).

Remark 1.16. We can write \mathcal{M}_{m_f} as a Cartesian product instead of a subset of a Cartesian product by replacing $h_j \in (0, \text{length}(\pi_2(\ell_{\Delta \cap \lambda_j})))$ with

$$\tilde{h}_j = \frac{h_j}{\text{length}(\pi_2(\Delta \cap \ell_{\lambda_j}))} \in (0, 1)$$

for $j = 1, \dots, m_f$. Then we have

$$\mathcal{M}_{m_f} \cong \mathcal{DPolyg}_{m_f}(\mathbb{R}^2) \times (0, 1)^{m_f} \times \mathbb{R}[[X, Y]]_0^{m_f}.$$

○

Theorem 1.17. *There exists a bijection between the set of simple semitoric integrable systems modulo semitoric isomorphism and \mathcal{M} , the set of semitoric lists of ingredients. In particular, for any nonnegative integer m_f we have that*

$$\mathcal{T}_{m_f} \cong \mathcal{M}_{m_f} \\ [(M, \omega, (H, J))] \leftrightarrow ([(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f}), (h_j)_{j=1}^{m_f}, ((S_j)^\infty)_{j=1}^{m_f}),$$

where the invariants $[(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f})]$, h_j , and $(S_j)^\infty$ are as defined above.

1.3. Main theorem. To define a metric on \mathcal{T} we will first define a metric on each invariant and then we will combine all of these metrics to form a metric on \mathcal{M} . Finally, we will pull this metric back by the map in Theorem 1.17 to produce a metric on the space of semitoric systems. This is the same strategy used in [27].

1.3.1. Comparing the Taylor series invariant. First we will define a metric on the Taylor series invariant. Let $\sum_{i,j \geq 0} \sigma_{i,j} X^i Y^j \in \mathbb{R}[[X, Y]]_0$ ¹³. From the construction in [18] we can see that the $\sigma_{0,1}$ term is only defined up to the addition of an element in $2\pi\mathbb{Z}$. This is why we have assumed that term is in $[0, 2\pi)$, but when defining the metric (in order to define the appropriate topology on semitoric systems) we must remember that as $\sigma_{0,1}$ is getting closer to 2π it is actually getting closer to zero as well. That is, we should think of $\sigma_{0,1}$ as an element of $\mathbb{R}/2\pi\mathbb{Z}$ and not as an element of $[0, 2\pi)$.

Definition 1.18. Suppose that $\{b_n\}_{n=0}^\infty$ is a sequence such that $b_n \in (0, \infty)$ for each $n \in \mathbb{Z}_{\geq 0}$ and $\sum_{n=0}^\infty n b_n < \infty$. We will say that such a sequence is *linear summable*. Now we define

$$d_0^{\{b_n\}_{n=0}^\infty} : \mathbb{R}[[X, Y]]_0 \times \mathbb{R}[[X, Y]]_0 \rightarrow \mathbb{R}$$

to be given by

$$d_0^{\{b_n\}_{n=0}^\infty} \left(\sum_{i,j \geq 0} \sigma_{i,j} X^i Y^j, \sum_{i,j \geq 0} \sigma'_{i,j} X^i Y^j \right) = \sum_{i,j \geq 0, (i,j) \neq (0,1)}^\infty \min\{|\sigma_{i,j} - \sigma'_{i,j}|, b_{i+j}\} \\ + \min\{|\sigma_{0,1} - \sigma'_{0,1}|, 2\pi - |\sigma_{0,1} - \sigma'_{0,1}|, b_1\}$$

¹³See Section 2.1 for the definition of a metric on $\mathbb{R}[[X, Y]]$ would could potentially be of independent interest.

where $\sum_{i,j \geq 0} \sigma_{i,j} X^i Y^j, \sum_{i,j \geq 0} \sigma'_{i,j} X^i Y^j \in \mathbb{R}[[X, Y]]_0$.

Remark 1.19. Notice that this metric places a higher weight on the lower order terms in the Taylor series, where the distribution of this weight is controlled by $\{b_n\}_{n=0}^\infty$. This means that two series which agree up to a high order will be very close in the metric space and two series which agree only on the high order terms will be distant. Of course, this is exactly what we would want in a metric on Taylor series. \oslash

The proof of the following Proposition is nearly identical to the proof of Proposition 2.2 in Section 2.1.

Proposition 1.20. *For any choice of linear summable sequence $\{b_n\}_{n=0}^\infty$ the space $(\mathbb{R}[[X, Y]]_0, d_0^{\{b_n\}_{n=0}^\infty})$ is a complete path-connected metric space and a sequence of Taylor series converges if and only if the coefficient of Y converges in $\mathbb{R}/2\pi\mathbb{Z}$ and all other terms converge in \mathbb{R} . Thus, the topology of $(\mathbb{R}[[X, Y]]_0, d_0^{\{b_n\}_{n=0}^\infty})$ does not depend on the choice of $\{b_n\}_{n=0}^\infty$.*

1.3.2. *Comparing the volume invariant.* Since the volume invariant h_j is a real number we can simply use the standard metric on \mathbb{R} . Thus we define

$$d_h(h, h') = |h - h'|.$$

Clearly this is a metric on any subset of \mathbb{R} .

1.3.3. *Comparing the affine invariant.* The topology of spaces of polygons have been studied by many authors. For example, in [12, 13] the authors study polygons with a fixed number of edges up to translations and positive homotheties in Euclidean space and in [15] the authors study polygons in \mathbb{R}^2 with fixed side length up to orientation preserving isometries. For this paper we will use a topology on polygons related to the Duistermaat-Heckman measure [6] similar to what is done in [27]. A natural way to define a metric on subsets of \mathbb{R}^2 (or any measurable space) is to use the volume of the symmetric difference¹⁴. Let $*$ denote the symmetric difference of sets. That is, for $A, B \in \mathbb{R}^2$ let

$$A * B = (A \setminus B) \cup (B \setminus A).$$

So in order to define a metric on labeled Delzant semitoric polygons we would like to use the volume of the symmetric difference of the polygons (as is done in [27]) but there are two problems. First, the polygons here are not required to be compact, so the symmetric difference may have infinite volume, and second there are many polygons to choose from. To solve the first problem we will define a non-standard measure on \mathbb{R}^2 ¹⁵. A natural choice would be a probability measure on \mathbb{R}^2 but the structure of $\mathcal{DPoly}_{m_f}(\mathbb{R}^2)$ is such that vertical translation should not affect the measure. This is because the elements of $\mathcal{DPoly}_{m_f}(\mathbb{R}^2)$ are only unique up to specific vertical transformations. With this in mind we make the following definition.

Definition 1.21. We say that a measure ν on \mathbb{R}^2 is *admissible* if:

¹⁴Of course, for the symmetric difference of sets to be a metric one must actually consider sets modulo measure zero corrections. We do not have to do this because the only sets we consider are polygons, and if the symmetric difference of two polygons has zero Lebesgue measure then they are equal.

¹⁵When only considering compact semitoric systems one can use the Lebesgue measure on \mathbb{R}^2 instead to produce a metric which induces the same topology, see Remark 2.16.

- (1) it is in the same measure class as μ , the Lebesgue measure on \mathbb{R}^2 (i.e. $\mu \ll \nu$ and $\nu \ll \mu$);
- (2) its Radon-Nikodym derivative with respect to Lebesgue measure only depends on the x -coordinate, i.e. there exists a $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $d\nu/d\mu(x, y) = g(x)$ for all $(x, y) \in \mathbb{R}^2$;
- (3) this function g satisfies $xg \in L^1(\mu, \mathbb{R})$ and g is bounded and bounded away from zero on any compact interval.

We say that a measurable map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a *vertical transformation* if the action of T does not change the x -coordinate of any point and vertical distances are invariant under T . That is, for any points $p_1, p_2 \in \mathbb{R}^2$ we have that $\pi_1 \circ T(p_1) = p_1$ and

$$\pi_2(p_1) - \pi_2(p_2) = \pi_2 \circ T(p_1) - \pi_2 \circ T(p_2).$$

Part 2 of Definition 1.21 implies that the measure is invariant under vertical transformations and part 3 will force the sets we are interested in (convex sets which have a finite height at every x -value) to have finite measure. There is an example of such a measure in Section 2.2.

Proposition 1.22. *Suppose that ν is an admissible measure on \mathbb{R}^2 and $\Delta \in \text{Polyg}(\mathbb{R}^2)$. Then Δ has everywhere finite height if and only if $\nu(\Delta) < \infty$.*

This Proposition is proven in Section 2.2. Let \mathcal{S}^{m_f} denote the symmetric group on m_f elements. For $p \in \mathcal{S}^{m_f}$ let the action of p on a vector $\vec{v} = (v_j)_{j=1}^{m_f}$ by permuting the elements be denoted by $p(\vec{v}) = (v_{p(j)})_{j=1}^{m_f}$.

Definition 1.23. Suppose $\vec{k}, \vec{k}' \in \mathbb{Z}^{m_f}$ for some nonnegative integer m_f . Then we say $\vec{k} \sim \vec{k}'$ if there exists a constant $c \in \mathbb{Z}$ and a permutation $p \in \mathcal{S}^{m_f}$ such that $k_j = k'_{p(j)} + c$ for all $j = 1, \dots, m_f$. We denote by $[\vec{k}]$ the equivalence class of \vec{k} in \mathbb{Z}^{m_f} / \sim .

Definition 1.24. Fix any $\vec{k}, \vec{k}' \in \mathbb{Z}^{m_f}$ such that $\vec{k} \sim \vec{k}'$. Let

$$\mathcal{S}_{\vec{k}, \vec{k}'}^{m_f} = \left\{ p \in \mathcal{S}^{m_f} \mid \begin{array}{l} \text{there exists } c \in \mathbb{Z} \text{ such that} \\ k_j = k'_{p(j)} + c \text{ for all } j = 1, \dots, m_f \end{array} \right\}.$$

Notice that $\vec{k} \sim \vec{k}'$ is equivalent to $\mathcal{S}_{\vec{k}, \vec{k}'}^{m_f} \neq \emptyset$. The elements of $\mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}$ will be called *appropriate permutations* for \vec{k} and \vec{k}' .

Now, assume that two labeled weighted polygons have the same number of focus-focus points and twisting indices related by \sim . We can shift the twisting index of one of the labeled weighted polygons by the action of an element of \mathcal{G} in such a way that such that after the shift the two labeled weighted polygons in question will have the same twisting index modulo the ordering. Once the twisting indices are fixed we still have a family of polygons which depends on the choice of $\vec{\epsilon} \in \{-1, +1\}^{m_f}$. The number of possible choices of $\vec{\epsilon}$ is finite so we will simply sum up the symmetric difference of each pair of polygons for each choice of $\vec{\epsilon}$. Using Remark 1.10 we can concisely write this as in the following definition.

Definition 1.25. Suppose that $[(\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})], [(\Delta', (\ell_{\lambda'_j}, +1, k'_j)_{j=1}^{m_f})] \in \mathcal{DPolyg}_{m_f}(\mathbb{R}^2)$ ¹⁶ for some $m_f > 0$ and with $\vec{k} \sim \vec{k}'$. For $p \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}$ define

$$d_P^{p, \nu}([(\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})], [(\Delta', (\ell_{\lambda'_j}, +1, k'_j)_{j=1}^{m_f})]) = \sum_{\vec{u} \in \{0,1\}^{m_f}} \nu(t_{\vec{u}}(\Delta) * t_{p(\vec{u})}(T^{-c}(\Delta')))$$

where $c \in \mathbb{Z}$ is defined to be the unique integer such that $k_j - k'_{p(j)} = c$ for all $j = 1, \dots, m_f$. In the case that $[(\Delta)], [(\Delta')] \in \mathcal{DPolyg}_0(\mathbb{R}^2)$ define¹⁷

$$d_P^{\nu}([(\Delta)], [(\Delta')]) = \nu(\Delta * \Delta').$$

Notice that this function is not a metric in general because it is not symmetric¹⁸ and recall such a p and related c must exist because $\vec{k} \sim \vec{k}'$. We will remove the dependence on a choice of permutation in the next section when we define the final version of the metric. Of course, there are many ways to choose a representative from each equivalence which have matching twisting indices (we can always act on both polygons by T^{c_2} , $c_2 \in \mathbb{Z}$), but the volume of the symmetric difference will not actually depend on that choice (see Proposition 2.5) so this function is well-defined on orbits of $G_{m_f} \times \mathcal{G}$.

1.3.4. *Defining the metric and stating the theorem.* An appropriate topology on \mathcal{T} would not allow a continuous way to change the number of singular points so it is reasonable to assume that systems with a different number of singular points would be in different components of \mathcal{T} . Additionally, since the invariant \vec{k} is discrete one might assume that different values of \vec{k} would not be comparable; this is not correct. In fact, if $\vec{k} \sim \vec{k}'$ then systems with these twisting indices can be compared via the metric we are about to define but they are in different connected components (Remark 2.19).

Definition 1.26. Suppose that $m_f \in \mathbb{Z}_{>0}$ and $\vec{k} \in \mathbb{Z}^{m_f}$. Then we define $\mathcal{T}_{m_f, \vec{k}} \subset \mathcal{T}_{m_f}$ to be those elements with twisting index exactly \vec{k} and define

$$\mathcal{T}_{m_f, [\vec{k}]} = \bigcup_{\vec{k}' \in [\vec{k}]} \mathcal{T}_{m_f, \vec{k}'}.$$

Furthermore, define

$$\mathcal{DPolyg}_{m_f, [\vec{k}]}(\mathbb{R}^2) = \{(\Delta, (\ell_{\lambda_j}, \epsilon_j, k'_j)_{j=1}^{m_f}) \in \mathcal{DPolyg}_{m_f}(\mathbb{R}^2) \mid \vec{k} \sim \vec{k}'\}$$

and

$$\mathcal{M}_{m_f, [\vec{k}]} = \mathcal{M}_{m_f} \cap \left(\mathcal{DPolyg}_{m_f, [\vec{k}]}(\mathbb{R}^2) \times \mathbb{R}^{m_f} \times \mathbb{R}[[X, Y]]_0^{m_f} \right).$$

Remark 1.27. Notice that

$$\mathcal{T} = \bigcup_{\substack{m_f \in \mathbb{Z}_{\geq 0} \\ \vec{k} \in \mathbb{Z}^{m_f}}} \mathcal{T}_{m_f, \vec{k}}.$$

¹⁶In fact, this function can actually be defined on all of $\mathcal{LWPolyg}_{m_f}(\mathbb{R}^2)$ without any changes. This will be used in Section 3.

¹⁷Since $m_f = 0$ the labeled weighted polygon becomes only a polygon and the group $G_0 \times \mathcal{G}$ is trivial so in fact it is a unique polygon. This should be thought of as the same formula as the $m_f > 0$ case and it is only treated separately because the sum in the more general formula would be empty if $m_f = 0$.

¹⁸The function $d_P^{p, \nu}$ is a metric if and only if $p = p^{-1}$ in \mathcal{S}^{m_f} .

This union, and the union in Definition 1.26, are not disjoint unions only because they have repeated terms. For instance, since the action of \mathcal{G} can shift all of the twisting indices, we have that

$$\mathcal{T}_{m_f, \vec{k}} = \mathcal{T}_{m_f, (k_j+c)_{j=1}^{m_f}}$$

for any $c \in \mathbb{Z}$. ⊙

From Sections 1.3.1, 1.3.2, and 1.3.3 given some fixed appropriate permutation we already know how to define a “distance” function on two systems with specified twisting index. To produce a metric which does not depend on fixing a permutation we will take the minimum of each possibility.

Definition 1.28. Let $m_f \in \mathbb{Z}_{\geq 0}$ and $\vec{k} \in \mathbb{Z}^{m_f}$ and suppose that $m, m' \in \mathcal{M}_{m_f, [\vec{k}]}$ with $m = ([\Delta_w], (h_j)_{j=1}^{m_f}, ((S_j)^\infty)_{j=1}^{m_f})$ and $m' = ([\Delta'_w], (h'_j)_{j=1}^{m_f}, ((S'_j)^\infty)_{j=1}^{m_f})$. Let ν be an admissible measure, $\{b_n\}_{n=0}^\infty$ be a linear summable sequence, and $p \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}$. We define:

(1) the *comparison with alignment* p to be

$$d_{m_f, \vec{k}}^{p, \nu, \{b_n\}_{n=0}^\infty}(m, m') = d_p^{p, \nu}([\Delta_w], [\Delta'_w]) + \sum_{j=1}^{m_f} \left(d_0^{\{b_n\}_{n=0}^\infty}((S_j)^\infty, (S'_{p(j)})^\infty) + d_h(h_j, h'_{p(j)}) \right);$$

(2) the *distance between* m and m' to be

$$d_{m_f, [\vec{k}]}^{\nu, \{b_n\}_{n=0}^\infty}(m, m') = \min_{p \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}} \{d_{m_f, \vec{k}}^{p, \nu, \{b_n\}_{n=0}^\infty}(m, m')\}.$$

A minimum of even a finite number of metrics is not a metric in general, but $d_{m_f, [\vec{k}]}^{\nu, \{b_n\}_{n=0}^\infty}$ is a metric in this case (see Section 1.3.4). Now we use this distance defined on each component to induce a distance on the whole space which can be pulled back to produce a metric on \mathcal{T} .

Definition 1.29. Let ν be an admissible measure and $\{b_n\}_{n=0}^\infty$ be a linear summable sequence. Then we define

(1) the *distance on* \mathcal{M} by

$$d^{\nu, \{b_n\}_{n=0}^\infty}(m, m') = \begin{cases} d_{m_f, [\vec{k}]}^{\nu, \{b_n\}_{n=0}^\infty}(m, m') & , \text{ if } m, m' \in \mathcal{M}_{m_f, [\vec{k}]} \text{ for some } m_f \in \mathbb{Z}, \vec{k} \in \mathbb{Z}^{m_f} \\ 1 & , \text{ otherwise} \end{cases}$$

for $m, m' \in \mathcal{M}$;

(2) the *distance on* \mathcal{T} by $\mathcal{D}^{\nu, \{b_n\}_{n=0}^\infty} = \Phi^* d^{\nu, \{b_n\}_{n=0}^\infty}$ where $\Phi : \mathcal{T} \rightarrow \mathcal{M}$ is the bijective correspondence from Theorem 1.17.

Of course any constant may be chosen for the value of $d^{\nu, \{b_n\}_{n=0}^\infty}$ in the case that the two lists of ingredients are incomparable. To state the main theorem we will have to first define the completion.

Definition 1.30. For any choice of $m_f \in \mathbb{Z}_{\geq 0}$ and $\vec{k} \in \mathbb{Z}^{m_f}$ we define

$$\widetilde{\mathcal{M}_{m_f, [\vec{k}]}} = \widetilde{\mathcal{DPolyg}_{m_f, [\vec{k}]}}(\mathbb{R}^2) \times [0, 1]^{m_f} \times \mathbb{R}[[X, Y]]_0^{m_f}$$

and

$$\tilde{\mathcal{M}} = \bigcup_{\substack{m_f \in \mathbb{Z}_{\geq 0} \\ \vec{k} \in \mathbb{Z}^{m_f}}} \widetilde{\mathcal{M}_{m_f, [\vec{k}]}}$$

where the critical points satisfy the ordering convention from Remark 3.8 and $\mathcal{DPoly}_{m_f, [\vec{k}]}(\mathbb{R}^2)$ is defined as in Definition 3.9.

Theorem A. *For any choice of*

- (1) *a linear summable sequence $\{b_n\}_{n=0}^\infty$;*
- (2) *an admissible measure ν*

we have that $(\mathcal{T}, \mathcal{D}^{\nu, \{b_n\}_{n=0}^\infty})$ is a non-complete metric space whose completion corresponds to $\tilde{\mathcal{M}}$. Moreover, the topology of $(\mathcal{T}, \mathcal{D}^{\nu, \{b_n\}_{n=0}^\infty})$ is independent of the choice of ν and $\{b_n\}_{n=0}^\infty$.

Remark 1.31. There are several important facts to notice about Theorem A.

- (1) This distance induces a unique topology on \mathcal{T} and thus Theorem A completely resolves Problem 2.43 from [25].
- (2) Since toric integrable systems fall into the broader category of semitoric systems it is natural to wonder if the metric defined in this paper is compatible with the metric on toric systems from [27]. Because we must choose an admissible measure to apply to the more general cases the metric induced by d does not exactly match the metric defined on toric systems but they do induce the same topology, see Section 2.5.
- (3) In special cases a less complicated form of the metric can be used. When studying only the compact semitoric systems the admissible measure on \mathbb{R}^2 can be instead replaced by the standard Lebesgue measure without changing the topology (Remark 2.16) and to study the topology of \mathcal{M} the metric d^{Id} may be used as is explained in Section 2.6 (Though it is clear that d produces the correct metric space structure on \mathcal{T} , see Remark 3.16).
- (4) Since all metric spaces are Tychonoff (completely regular and Hausdorff) we know that \mathcal{T} is Tychonoff. Thus the Stone-C ech compactification [30, 29] applies to \mathcal{T} so it admits a Hausdorff compactification (just as in [27]).
- (5) Since an integrable system is just a manifold and a map into \mathbb{R}^n for some $n \in \mathbb{N}$ one may consider using a general metric on maps to define a metric on integrable systems. A metric defined on collections of maps with varying domains as is in [20], while very general, would actually not be appropriate in this situation because the singularities, and thus isomorphism type, of semitoric systems can be changed by perturbing the systems on arbitrarily small sets. For instance, this can be seen because the Taylor series invariant is completely independent of the other invariants. For this reason we have defined a metric on the invariants, which describe the essential properties of the integrable system, to produce an appropriate metric on \mathcal{T} .

⊗

2. THE METRIC

In this section we fill in the details of constructing the metric and prove that it is a metric.

2.1. Metrics on Taylor series. Let $\mathbb{R}[[X, Y]]$ refer to the algebra of real formal power series in two variables.

Definition 2.1. Suppose that $\{b_n\}_{n=0}^\infty$ is any linear summable sequence. Then we define the *distance on Taylor series* to be the function

$$d_{\mathbb{R}[[X, Y]]}^{\{b_n\}_{n=0}^\infty} : \mathbb{R}[[X, Y]] \times \mathbb{R}[[X, Y]] \rightarrow \mathbb{R}$$

given by

$$d_{\mathbb{R}[[X, Y]]}^{\{b_n\}_{n=0}^\infty} \left(\sum_{i,j \geq 0} \sigma_{i,j} X^i Y^j, \sum_{i,j \geq 0} \sigma'_{i,j} X^i Y^j \right) = \sum_{i,j=0}^\infty \min \{ |\sigma_{i,j} - \sigma'_{i,j}|, b_{i+j} \}$$

where $\sum_{i,j \geq 0} \sigma_{i,j} X^i Y^j, \sum_{i,j \geq 0} \sigma'_{i,j} X^i Y^j \in \mathbb{R}[[X, Y]]$.

Proposition 2.2. *The space $(\mathbb{R}[[X, Y]], d_{\mathbb{R}[[X, Y]]}^{\{b_n\}_{n=0}^\infty})$ is a complete path-connected metric space and a sequence of Taylor series converges if and only if each term converges.*

Proof. First notice that the sum in the definition of the distance always converges. This is clear because

$$d_{\mathbb{R}[[X, Y]]}^{\{b_n\}_{n=0}^\infty} \left(\sum_{i,j \geq 0} \sigma_{i,j} X^i Y^j, \sum_{i,j \geq 0} \sigma'_{i,j} X^i Y^j \right) \leq \sum_{i,j=0}^\infty b_{i+j} = \sum_{n=0}^\infty (n+1)b_n < \infty$$

for any $\sum_{i,j \geq 0} \sigma_{i,j} X^i Y^j, \sum_{i,j \geq 0} \sigma'_{i,j} X^i Y^j \in \mathbb{R}[[X, Y]]$ by the choice of $\{b_n\}_{n=0}^\infty$. It is also clear that $d_{\mathbb{R}[[X, Y]]}^{\{b_n\}_{n=0}^\infty}$ is symmetric and positive definite. It satisfies the triangle inequality because that inequality is satisfied for each term and thus we can see that $(\mathbb{R}[[X, Y]], d_{\mathbb{R}[[X, Y]]}^{\{b_n\}_{n=0}^\infty})$ is a metric space.

Next we will prove the condition on convergence. Suppose that

$$\lim_{k \rightarrow \infty} d_{\mathbb{R}[[X, Y]]}^{\{b_n\}_{n=0}^\infty} \left(\sum_{i,j \geq 0} \sigma_{i,j}^k X^i Y^j, \sum_{i,j \geq 0} \sigma_{i,j}^0 X^i Y^j \right) = 0$$

with $\sum_{i,j \geq 0} \sigma_{i,j}^k X^i Y^j, \sum_{i,j \geq 0} \sigma_{i,j}^0 X^i Y^j \in \mathbb{R}[[X, Y]]$ for $k \in \mathbb{Z}_{\geq 0}$. Fix any $I, J \in \mathbb{Z}_{\geq 0}$ and we will show that $\sigma_{I,J}^k \rightarrow \sigma_{I,J}^0$ in \mathbb{R} as $k \rightarrow \infty$. Fix $\varepsilon > 0$ and find K such that $k > K$ implies that

$$\sum_{i,j=0}^\infty \min \{ |\sigma_{i,j}^k - \sigma_{i,j}^0|, b_{i+j} \} < \varepsilon$$

because we may assume that $\varepsilon < b_{I+J}$. Then we can see that $|\sigma_{I,J}^k - \sigma_{I,J}^0| < \varepsilon$ so the result follows.

Now we will show the converse. Suppose that

$$\lim_{k \rightarrow \infty} |\sigma_{i,j}^k - \sigma_{i,j}^0|$$

for all $i, j \in \mathbb{Z}_{\geq 0}$. Fix $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that

$$\sum_{n \geq N} (n+1)b_n < \varepsilon/2,$$

and let $K \in \mathbb{Z}$ be such that $k > K$ implies that

$$|\sigma_{i,j}^k - \sigma_{i,j}^0| < \frac{\varepsilon}{N(N+1)}$$

for each $i, j \in \mathbb{Z}_{\geq 0}$ such that $i + j < N$. Notice it is possible to do this simultaneously because there are only finitely many such pairs (i, j) . Now we can see that for any $k > K$ we have that

$$\begin{aligned} d_{\mathbb{R}[[X,Y]]}^{\{b_n\}_{n=0}^\infty} \left(\sum_{i,j \geq 0} \sigma_{i,j}^k X^i Y^j, \sum_{i,j \geq 0} \sigma_{i,j}^0 X^i Y^j \right) &\leq \sum_{i+j < N} |\sigma_{i,j}^k - \sigma_{i,j}^0| + \sum_{i+j \geq N} b_{i+j} \\ &< \frac{\varepsilon}{N(N+1)} \sum_{i+k < N} 1 + \sum_{n \geq N} (n+1)b_n \\ &< \frac{\varepsilon}{N(N+1)} \frac{N(N+1)}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus we have proven the convergence condition.

Since any element of this space may be continuously transformed into any other as a linearly in each term it is path-connected so to finish the proof we will show that this space is complete. Suppose that $\left(\sum_{i,j \geq 0} \sigma_{i,j}^k X^i Y^j \right)_{k=0}^\infty$ is a Cauchy sequence in $\mathbb{R}[[X, Y]]$. Similar to the argument for convergence we can see that this means that $\{\sigma_{i,j}^k\}_{k=0}^\infty$ is Cauchy for each fixed pair of values $i, j \in \mathbb{Z}_{\geq 0}$. Since $[0, \infty)$ is complete we conclude that for each $i, j \in \mathbb{Z}_{\geq 0}$ there exists some $\sigma_{i,j}^0$ such that $\sigma_{i,j}^k \rightarrow \sigma_{i,j}^0$ as $k \rightarrow \infty$ in \mathbb{R} . Since it converges in each term, we can use the convergence condition to conclude that

$$\lim_{k \rightarrow \infty} d_{\mathbb{R}[[X,Y]]}^{\{b_n\}_{n=0}^\infty} \left(\sum_{i,j \geq 0} \sigma_{i,j}^k X^i Y^j, \sum_{i,j \geq 0} \sigma_{i,j}^0 X^i Y^j \right) = 0$$

and so all Cauchy sequences have limits. \square

Now we have characterized convergence in this space in a way which is independent of the sequence $\{b_n\}_{n=0}^\infty$. Thus we have the following result.

Corollary 2.3. *The topology on $\mathbb{R}[[X, Y]]$ determined by $d_{\mathbb{R}[[X,Y]]}^{\{b_n\}_{n=0}^\infty}$ does not depend on the choice of linear summable sequence $\{b_n\}_{n=0}^\infty$.*

Notice that Remark 1.19 also applies in this case, so we have produced an appropriate metric on Taylor series in two variables¹⁹. Also notice that $\mathbb{R}[[X, Y]]_0$ is not a closed subset of $(\mathbb{R}[[X, Y]], d_{\mathbb{R}[[X,Y]]}^{\{b_n\}_{n=0}^\infty})$ and in fact $(\mathbb{R}[[X, Y]]_0, d_{\mathbb{R}[[X,Y]]}^{\{b_n\}_{n=0}^\infty})$ with the restricted metric is not a complete metric space. To see this simply consider any collection of Taylor series in which $\sigma_2 \rightarrow 2\pi$. This does not accurately describe the structure of the semitoric systems and thus we use the altered metric from Definition 1.18. The proof of Proposition 2.2 also proves Proposition 1.20 with very few changes.

Remark 2.4. A similar construction to $d_{\mathbb{R}[[X,Y]]}^{\{b_n\}_{n=0}^\infty}$ can be used to produce such a metric on Taylor series in any number of variables. The only difference is that to produce a metric on

¹⁹A similar construction can be used for Taylor series on more variables. See Remark 2.4

Taylor series in m variables the sequence $\{b_n\}_{n=0}^\infty$ would be required to satisfy

$$\sum_{n=0}^{\infty} \binom{n+m-1}{n} b_n < \infty$$

because there are $\binom{n+m-1}{n}$ terms of degree n in a Taylor series on m variables. \oslash

2.2. Metrics on labeled weighted polygons. We start this section with a proof.

Proof of Proposition 1.22. By definition Δ is the intersection of half-spaces and by since it is assumed to have everywhere finite height we can see that this collection of half spaces must include at least two which are not completely vertical (i.e. not of the form $\{x \geq c\}$ or $\{x \leq c\}$ for $c \in \mathbb{R}$). Let B denote the intersection of these two half planes. Then by definition $\Delta \subset B$ and thus $\nu(\Delta) < \nu(B)$. If the two half planes are parallel of a distance c apart then

$$\nu(B) = \int_B \frac{d\nu}{d\mu} d\mu = \int_{\mathbb{R}} cg d\mu < \infty$$

because $xg \in L^1(\mu, \mathbb{R})$ implies that $g \in L^1(\mu, \mathbb{R})$. If then spaces are not parallel then their boundaries intersect at some point (x_0, y_0) . Let m be the absolute value of the difference in the slopes of the two boundaries. Then for each value $x \in \mathbb{R}^2$ the height of B at that x -coordinate is $m|x - x_0|$ and the sign of $x - x_0$ is the same for each $(x, y) \in B$. Without loss of generality assume that $x - x_0 \geq 0$ for all $(x, y) \in B$ so we have

$$\nu(B) = \int_B \frac{d\nu}{d\mu} d\mu = \int_{x_0}^{\infty} m(x - x_0)g(x) d\mu = m \int_{x_0}^{\infty} xg d\mu - mx_0 \int_{x_0}^{\infty} g d\mu < \infty$$

because $g \in L^1(\mu, \mathbb{R})$ and $xg \in L^1(\mu, \mathbb{R})$. The computation is similar if $x - x_0 \leq 0$ for all $(x, y) \in B$.

Any compact set without everywhere finite height will have infinite ν -measure. This is because a compact set which does not have everywhere finite height either is a vertical line, which is not a polygon, or includes a subset of the form $\{(x, y) \mid a_1 < x < a_2\}$ for some $a_1 < a_2$. Such a subset has infinite ν -measure because ν is invariant under vertical translations. \square

Even once we have fixed the cut directions there are many polygons to choose from based on the choice of the twisting index (i.e. the orbit of the action of \mathcal{G}) but so long as the same choice is made for each pair of polygons this choice does not change the volume of the symmetric difference.

Proposition 2.5. Let $m_f \in \mathbb{Z}_{\geq 0}$, $p \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}$, and let $\mathcal{J}^p \subset (\mathcal{DPolyg}_{m_f, [\vec{k}]}(\mathbb{R}^2))^2$ be given by

$$\mathcal{J}^p = \{([\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f}], [(\Delta', (\ell'_{\lambda'_j}, \epsilon'_j, k'_j)_{j=1}^{m_f})]) \in (\mathcal{DPolyg}_{m_f, [\vec{k}]}(\mathbb{R}^2))^2 \mid p \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}\}.$$

Then the function $d_P^{p, \nu} : \mathcal{J}^p \rightarrow \mathbb{R}$ is well defined.

Proof. Suppose that

$$\Delta_w^1 = (\Delta^1, (\ell_{\lambda_j}^1, +1, k_j^1)_{j=1}^{m_f}), \Delta_w^2 = (\Delta^2, (\ell_{\lambda_j}^2, +1, k_j^2)_{j=1}^{m_f}) \in [(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f})]$$

and $\Delta'_w = (\Delta', (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f}) \in [(\Delta', (\ell_{\lambda'_j}, \epsilon'_j, k'_j)_{j=1}^{m_f})]$. Then there exists some $d \in \mathbb{Z}$ such that $k_j^1 = k_j^2 - d$ for $j = 1, \dots, m_f$ and $\Delta^1 = T^d(\Delta^2)$. Find $c \in \mathbb{Z}$ such that $k_j^1 - k'_{p(j)} = c$ and notice that this means that $k_j^2 - k'_{p(j)} = c + d$. Now we have

$$\begin{aligned} d_P^{p,\nu}([\Delta_w^1], [\Delta'_w]) &= \sum_{\vec{u} \in \{0,1\}^{m_f}} \nu(t_{\vec{u}}(\Delta^1) * t_{p(\vec{u})}(T^{-c}(\Delta'))) \\ &= \sum_{\vec{u} \in \{0,1\}^{m_f}} \nu(t_{\vec{u}}(T^d(\Delta^2)) * t_{p(\vec{u})}(T^{-c}(\Delta'))) \\ &= \sum_{\vec{u} \in \{0,1\}^{m_f}} \nu(T^d(t_{\vec{u}}(\Delta^2) * t_{p(\vec{u})}(T^{-c-d}(\Delta')))) \\ &= \sum_{\vec{u} \in \{0,1\}^{m_f}} \nu(t_{\vec{u}}(\Delta^2) * t_{p(\vec{u})}(T^{-(c+d)}(\Delta'))) \\ &= d_P^{p,\nu}([\Delta_w^2], [\Delta'_w]) \end{aligned}$$

because admissible measures are invariant under vertical transformations such as T^d . The argument that this function is well defined in the second input is similar. \square

An example of an admissible measure on \mathbb{R}^2 is the following. Define ν_0 so that

$$\frac{d\nu_0}{d\mu}(x, y) = \begin{cases} 1 & , \text{ if } |x| < 1 \\ \frac{1}{x^3} & , \text{ else.} \end{cases}$$

The following proposition is obvious because $x^{-2} \in L^1(\mu, \mathbb{R})$ and $g_0 = \frac{d\nu_0}{d\mu}(x, 0)$ is bounded and bounded away from zero on compact intervals.

Proposition 2.6. *The measure ν_0 is an admissible measure on \mathbb{R}^2 .*

2.3. Choice of ν does not change the topology. It is important that while the choice of admissible measure will change the metric it does not change the topology induced by that metric. First we have Lemma 2.7 which will be used below to prove Lemma 2.8 and will also be used in Section 2.5.

Lemma 2.7. *Suppose that ν is an admissible measure and that $\Delta_k, \Delta \in \text{Polyg}(\mathbb{R}^2)$ for $k \in \mathbb{N}$ such that $\nu(\Delta_k * \Delta) \rightarrow 0$ as $k \rightarrow \infty$. Then there exists $x_0, y_0, y_1 \in \mathbb{R}$ and $K > 0$ such that $y_0 < y_1$ and $k > K$ implies that $A_{x_0, y_0, y_1} \subset \Delta_k \cap \Delta$ where $A_{x_0, y_0, y_1} = \{(x, y) \in \mathbb{R}^2 \mid y \in [y_0, y_1]\}$.*

Proof. Fix any $N > 0$ such that $\{(x, y) \in \Delta \mid x \in [-N, +N]\}$ has non-zero measure with respect to ν (and thus also with respect to μ). Since ν is admissible we can find some $c > 0$ such that $d\nu/d\mu > c$ on $[-N, N] \times \mathbb{R}$.

For each $\varepsilon > 0$ let $U_\varepsilon = \{p \in (-N, N) \times \mathbb{R} \mid B(\sqrt{2c\varepsilon}/\pi, p) \subset \Delta\}$ where $B(r, p)$ is the standard ball of radius $r \geq 0$ centered at $p \in \mathbb{R}^2$. Now let H be the intersection of $B(\sqrt{2c\varepsilon}/\pi, p)$ with any open half-plane with boundary through p . Then

$$\nu(H) = \frac{1}{c} \mu(H) \geq \frac{1}{2c} \mu(B(\sqrt{2c\varepsilon}/\pi, p)) = \frac{1}{2c} \pi \left(\sqrt{\frac{2c\varepsilon}{\pi}} \right)^2 = \varepsilon.$$

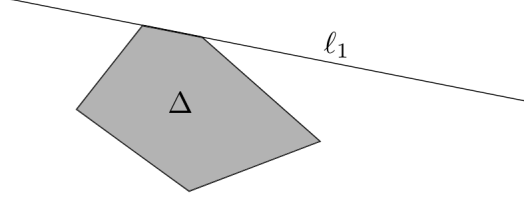


FIGURE 2. Since Δ is convex it must all lie on the same side of ℓ_1 .

Now, fix any $k \in \mathbb{N}$. We know that Δ_k is the intersection of closed half-planes which means that its complement Δ_k^c is the union of open half-planes. If $q \in U_\varepsilon \setminus \Delta_k$ that means that there exists some open half-plane with boundary including q which is a subset of Δ_k^c . Let H be the intersection of this half-plane with $B(\sqrt{2c\varepsilon}/\pi, q)$. Then, as above, $\nu(H) \geq \varepsilon$ and now $H \subset \Delta_k * \Delta$. So we conclude that $U_\varepsilon \setminus \Delta_k$ non-empty implies that $\nu(\Delta_k * \Delta) \geq \varepsilon$.

Now choose ε small enough that U_ε is non-empty and choose $K > 0$ such that $k > K$ implies that $\nu(\Delta * \Delta_k) < \varepsilon$. Thus $U_\varepsilon \setminus \Delta_k$ must be empty to avoid a contradiction and we conclude that $U_\varepsilon \subset \Delta_k$ for $k > K$. Clearly $U_\varepsilon \subset \Delta_k$ is open so we can find the set A_{x_0, y_0, y_1} as given in the statement of the Lemma. \square

Now we will use Lemma 2.7 to prove Lemma 2.8, which says that the same sequences of polygons converge with respect to any admissible measure.

Lemma 2.8. *Suppose that ν_1, ν_2 are admissible measures and that $\Delta_k, \Delta \in \text{Poly}(\mathbb{R}^2)$ for $k \in \mathbb{N}$ have $\nu_1(\Delta), \nu_1(\Delta_k) < \infty$. If $\nu_1(\Delta_k * \Delta) \rightarrow 0$ as $k \rightarrow \infty$ then $\nu_2(\Delta_k * \Delta) \rightarrow 0$ as $k \rightarrow \infty$.*

Proof. Suppose that $\nu_1(\Delta_k * \Delta) \rightarrow 0$ as $k \rightarrow \infty$ and let A, x_0, y_0 , and y_1 be as in Lemma 2.7. We know that the line $\{x = x_0\}$ intersects Δ so it must intersect the top boundary of Δ (since Δ has everywhere finite height by Proposition 1.22). Since a convex set is the intersection of half-planes there must exist a line ℓ_1 which goes through the point where $\{x = x_0\}$ intersects the top boundary such that all of Δ is in a closed half-plane bounded by ℓ_1 (as in Figure 2). Such a line may not be unique if there is a vertex with x -coordinate equal to x_0 , but any choice of such a line will do.

Let m denote the slope of ℓ_1 and let ℓ_2 be the line through (x_0, y_1) with slope $m + 1$. Let m' denote the slope of the line through the point (x_0, y_0) and the point which is the intersection of ℓ_1 with ℓ_2 . Finally let ℓ_3 be the line through (x_0, y_0) with slope $(m+m')/2$. Since the slope of ℓ_3 is greater than the slope of ℓ_2 these two lines must intersect at some x -coordinate greater than x_0 , but since the slope of ℓ_3 is less than m' we know that the intersection of ℓ_2 and ℓ_3 must have a greater x -coordinate than the intersection of ℓ_1 and ℓ_2 . Thus the lines ℓ_1, ℓ_2 , and ℓ_3 bound a triangle which we will denote by G , as is shown in Figure 3. Let $N_1 = \max_{s \in G} \pi_1(s)$. Since Δ is on one side of ℓ_1 and G is on the other we conclude that $G \cap \Delta = \emptyset$.

For any $N \in \mathbb{R}$ let E_N^1 denote the region of \mathbb{R}^2 which has $x > N > x_0$ and is above or on ℓ_2 .

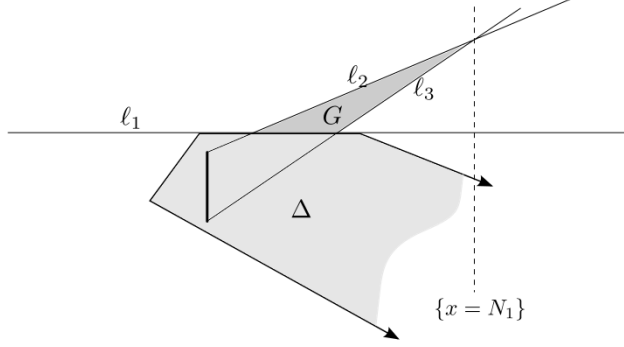


FIGURE 3. The lines ℓ_1, ℓ_2, ℓ_3 , and triangle G .

Now suppose that k is large enough so that $A \subset \Delta_k$. Suppose $p \in E_{N_1}^1$ and $p \in \ell_2$. Then $p \in \Delta_k$ implies that $G \subset \Delta_k * \Delta$ because Δ_k is convex and $\Delta \cap G = \emptyset$. Similarly, if p is any other point in $E_{N_1}^1$ we can conclude that some ν_1 -preserving transformation of G must be contained in $\Delta_k * \Delta$. This is because moving p vertically will result in acting on G by some matrix T^r (as in Equation (1)) with $r \in \mathbb{R}$ with origin on the line $\{x = x_0\}$ (see Figure 4). In any case, if $\Delta_k \cap E_{N_1}^1$ is nonempty and k is large enough so that $A \subset \Delta_k$ then we can conclude that $\nu_1(\Delta * \Delta_k) \geq \nu_1(G) > 0$. Since $\nu_1(\Delta * \Delta_k) \rightarrow 0$ as $k \rightarrow \infty$ we can conclude that for large enough k the set $\Delta_k \cap E_N^1$ is empty.

Using a similar argument, one can define sets E_N^i for $i = 2, 3, 4$ that must also be disjoint from Δ_k for large enough k and N . The sets E_N^1 and E_N^2 are bounded to the left by the line $\{x = N\}$ and the sets E_N^3 and E_N^4 are bounded to the right by $\{x = -N\}$. The sets E_N^1 and E_N^4 are bounded below by lines and the sets E_N^2 and E_N^3 are bounded above by lines. Let $E_N = \bigcap_{i=1}^4 E_N^i$ and let $N_2 > N_1$ be large enough so that for large enough k we have that $\Delta_k \cap E_{N_2} = \emptyset$. Let $D_N = [-N, N] \times \mathbb{R}$ for $N \in \mathbb{R}$ and let $S_N = \mathbb{R}^2 \setminus (E_N \cup D_N)$. This whole situation is shown in Figure 5.

Fix $\varepsilon > 0$. Notice that for each $N > 0$ the set S_N is of finite ν_2 -measure. Since $\{S_N\}_{N>0}$ are nested we conclude that $\lim_{N \rightarrow \infty} \nu_2(S_N) = 0$. Now choose some fixed $N_3 > N_2$ and

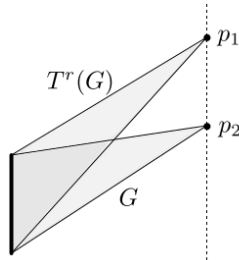


FIGURE 4. Notice that for a fixed vertical line segment $A \subset \mathbb{R}^2$ the measure of the convex hull of A and $p \in \mathbb{R}^2$ only depends on the x -component of p . This is because if $p_1, p_2 \in \mathbb{R}^2$ with $\pi_1(p_1) = \pi_1(p_2)$ then the convex hulls are related by a vertical transformation.

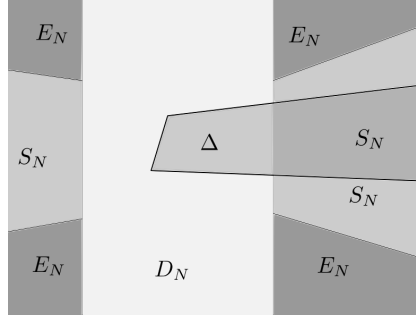


FIGURE 5. For large choices of N and k the set S_N is small and the set E_N has empty intersection with Δ_k . Then we can concentrate on the set D_N , on which the Radon-Nikodym derivative $d\nu_2/d\nu_1$ is bounded.

$K_1 > 0$ such that $\nu_2(S_{N_3}) < \varepsilon/2$ and $k > K_1$ implies that $\Delta_k \cap E_{N_3} = \emptyset$. Since both ν_1 and ν_2 are admissible measures we know that their Radon-Nikodym derivative is bounded on D_{N_3} . This is because

$$\frac{d\nu_2}{d\nu_1} = \frac{d\nu_2}{d\mu} \left(\frac{d\nu_1}{d\mu} \right)^{-1},$$

which are both bounded on D_{N_3} . Let $c > 0$ be such that $d\nu_2/d\nu_1 < c$ on D_{N_3} . Now choose $K_2 > K_1$ such that $k > K_2$ implies $\nu_1(\Delta * \Delta_k) < \varepsilon/2c$. Finally, for $k > K_2$ we have

$$\begin{aligned} \nu_2(\Delta_k * \Delta) &= \int_{\mathbb{R}^2} |\chi_{\Delta_k} - \chi_{\Delta}| d\nu_2 \\ &= \int_{S_{N_3}} |\chi_{\Delta_k} - \chi_{\Delta}| d\nu_2 + \int_{E_{N_3}} |\chi_{\Delta_k} - \chi_{\Delta}| d\nu_2 + \int_{D_{N_3}} |\chi_{\Delta_k} - \chi_{\Delta}| d\nu_2 \\ &\leq \nu_2(S_N) + 0 + \int_{D_{N_3}} |\chi_{\Delta_k} - \chi_{\Delta}| \frac{d\nu_2}{d\nu_1} d\nu_1 \\ &< \varepsilon/2 + c \nu_1(\Delta_k * \Delta) \\ &< \varepsilon/2 + c \varepsilon/2c = \varepsilon. \end{aligned}$$

□

By combining Lemma 2.8 and Proposition 1.20 we have the following corollary.

Corollary 2.9. *Fix a nonnegative integer $m_f \in \mathbb{Z}_{\geq 0}$, a vector $\vec{k} \in \mathbb{Z}^{m_f}$, any two linearly summable sequences $\{b_n\}_{n=0}^{\infty}$ and $\{b'_n\}_{n=0}^{\infty}$, and two admissible measures ν and ν' . Then the metric spaces $(\mathcal{M}_{m_f, [\vec{k}]}, d_{m_f, [\vec{k}]}^{\nu, \{b_n\}_{n=0}^{\infty}})$ and $(\mathcal{M}_{m_f, [\vec{k}]}, d_{m_f, [\vec{k}]}^{\nu', \{b'_n\}_{n=0}^{\infty}})$ have the same topology generated by their respective metrics.*

2.4. d is a metric. While it does not hold in general that the minimum of even a finite collection of metrics will be itself a metric, it does hold in this particular case because of the structure of our metric. For this section fix an admissible measure ν , a linear summable sequence $\{b_n\}_{n=0}^{\infty}$, a nonnegative integer m_f , and $\vec{k}, \vec{k}' \in \mathbb{Z}^{m_f}$. Let d denote $d_{m_f, [\vec{k}]}^{\nu, \{b_n\}_{n=0}^{\infty}}$ and let d^p denote $d_{m_f, \vec{k}}^{p, \nu, \{b_n\}_{n=0}^{\infty}}$. It is clear that d is positive definite and it is symmetric because $\mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}$ is closed under inverses so we must only show that the triangle inequality holds. We show this in Lemma 2.12 but first we must prove two lemmas.

Lemma 2.10. Fix $\vec{k}, \vec{k}', \vec{k}'' \in \mathbb{Z}^{m_f}$ and let $\mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}$ be as in Definition 1.24. Then for any fixed $q \in \mathcal{S}_{\vec{k}, \vec{k}''}^{m_f}$ we have that $\mathcal{S}_{\vec{k}'', \vec{k}'}^{m_f} = \{p \circ q^{-1} | p \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}\}$.

Proof. Let $r \in \mathcal{S}_{\vec{k}'', \vec{k}'}^{m_f}$. We know there exist constants $c_1, c_2 \in \mathbb{Z}$ such that

$$k_j - k''_{q(j)} = c_1 \text{ and } k''_j - k'_{r(j)} = c_2$$

for all $j = 1, \dots, m_f$. Fix some such j and let $i = q(j)$. This means in particular that

$$k_{q^{-1}(i)} = c_1 + k''_i \text{ and } k'_{r(i)} = k''_i - c_2.$$

Now using all of this information we can see that

$$\begin{aligned} k_j - k'_{r(q(j))} &= k_{q^{-1}(i)} - k'_{r(i)} \\ &= (c_1 + k''_i) - (k''_i - c_2) \\ &= c_1 + c_2 \end{aligned}$$

and so we conclude that $p = r \circ q \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}$ and clearly $r = p \circ q^{-1}$ so $\mathcal{S}_{\vec{k}'', \vec{k}'}^{m_f} \subset \{p \cdot q^{-1} | p \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}\}$.

Now let $p \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}$ and $q \in \mathcal{S}_{\vec{k}, \vec{k}''}^{m_f}$ so there must be constants $c, c_1 \in \mathbb{Z}$ such that

$$k_j - k'_{p(j)} = c \text{ and } k_j - k''_{q(j)} = c_1.$$

Subtracting these two equations gives $k''_{q(j)} - k'_{p(j)} = c - c_1$ and thus $p \circ q^{-1} \in \mathcal{S}_{\vec{k}'', \vec{k}'}^{m_f}$. \square

Lemma 2.11. Let $m, m', m'' \in \mathcal{M}_{m_f, [\vec{k}]}$ and suppose $p \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}$ and $q \in \mathcal{S}_{\vec{k}, \vec{k}''}^{m_f}$. Then

$$d^p(m, m') \leq d^q(m, m'') + d^{p \circ q^{-1}}(m'', m').$$

Proof. The $m_f = 0$ case is trivial so assume $m_f > 0$. Since $p \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}$ and $q \in \mathcal{S}_{\vec{k}, \vec{k}''}^{m_f}$ there must be constants $c, c_1 \in \mathbb{Z}$ such that

$$k_j - k'_{p(j)} = c \text{ and } k_j - k''_{q(j)} = c_1.$$

Since d^p is a sum of distances use the triangle inequality for each term with a appropriate permutation on the elements. That is,

$$\begin{aligned} d^p(m, m') &= \sum_{\vec{u} \in \{0,1\}^{m_f}} \nu(t_{\vec{u}}(\Delta) * t_{p(\vec{u})}(T^{-c}(\Delta'))) \\ &\quad + \sum_{j=1}^{m_f} \left(d_0^{\{b_n\}_{n=0}^\infty}((S_j)^\infty, (S'_{p(j)})^\infty) + d_h(h_j - h'_{p(j)}) \right) \\ &\leq \sum_{\vec{u} \in \{0,1\}^{m_f}} \left[\nu(t_{\vec{u}}(\Delta) * t_{q(\vec{u})}(T^{-c_1}(\Delta''))) + \nu(t_{q(\vec{u})}(T^{-c_1}(\Delta'')) * t_{p(\vec{u})}(T^{-c}(\Delta'))) \right] \\ &\quad + \sum_{j=1}^{m_f} \left(d_0^{\{b_n\}_{n=0}^\infty}((S_j)^\infty, (S''_{q(j)})^\infty) + d_0^{\{b_n\}_{n=0}^\infty}((S''_{q(j)})^\infty, (S'_{p(j)})^\infty) \right. \\ &\quad \left. + d_h(h_j - h''_{q(j)}) + d_h(h''_{q(j)} - h'_{p(j)}) \right) \\ &= d^q(m, m'') + d^{p \circ q^{-1}}(m'', m'). \end{aligned}$$

\square

Notice that in the case that $p = q = \text{Id}$ this gives a proof of the triangle inequality for d^{Id} . Now to show that the triangle inequality holds for d .

Lemma 2.12. *The triangle inequality holds for d .*

Proof. Let $m, m', m'' \in \mathcal{M}_{m_f, [\vec{k}]}$. There exists some $q \in \mathcal{S}_{\vec{k}, \vec{k}''}^{m_f}$, such that $d(m, m'') = d^q(m, m'')$ and by Lemma 2.10 we know that

$$\min_{p \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}} \{d^{p \circ q^{-1}}(m'', m')\} = d(m'', m').$$

Now, using the inequality from Lemma 2.11 we have that

$$\begin{aligned} d(m, m') &= \min_{p \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}} \{d^p(m, m')\} \\ &\leq \min_{p \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}} \{d^q(m, m'') + d^{p \circ q^{-1}}(m'', m')\} \\ &= d^q(m, m'') + \min_{p \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}} \{d^{p \circ q^{-1}}(m'', m')\} \\ &= d(m, m'') + d(m'', m') \end{aligned}$$

as desired. \square

Combining the arguments in Sections 2.1 and 2.2 with the present section, in particular Proposition 1.20 and Lemma 2.12, we get the following.

Proposition 2.13. *Let $m_f \in \mathbb{Z}_{\geq 0}$, $\vec{k} \in \mathbb{Z}^{m_f}$, $\{b_n\}_{n=0}^\infty$ be a linear summable sequence, and ν an admissible measure. Then the space $(\mathcal{M}_{m_f, [\vec{k}]}, d_{m_f, [\vec{k}]}^{\nu, \{b_n\}_{n=0}^\infty})$ is a metric space.*

2.5. Relation to the metric on the moduli space of toric systems. In [27] the authors construct a metric on the space of (compact) toric integrable systems which we denote $\mathcal{T}_{\mathbb{T}}$. Recall there is a one-to-one correspondence between elements of $\mathcal{T}_{\mathbb{T}}$ and Delzant polytopes (see Definition 1.8). The authors of [27] define a metric on $\mathcal{T}_{\mathbb{T}}$ by pulling back the natural metric on the space of Delzant polytopes given by the Lebesgue measure of the symmetric difference.

Now consider semitoric systems with no focus-focus singularities. If $m_f = 0$ then the set $G_{m_f} \times \mathcal{G}$ is empty and thus the affine invariant is a unique polygon, the Delzant polytope. To compare two such systems the semitoric metric defined in the present paper takes the ν -measure of the symmetric difference of the polygons for some admissible measure ν , as opposed to using the standard Lebesgue measure on \mathbb{R}^2 as is done in [27]. Notice also that $\mathcal{T}_{\mathbb{T}}$ is not equal to \mathcal{T}_0 because, for instance, there are elements of \mathcal{T}_0 which are not compact.

Moreover it is possible for two toric systems to be isomorphic as semitoric systems but not isomorphic as toric systems. This is because if $(M, \omega, (J, H))$ and $(M', \omega', (J', H'))$ are two choices of 4 dimensional toric systems then a diffeomorphism $\phi : M \rightarrow M'$ is an isomorphism of toric systems if $\phi^*(J', H') = (J, H)$. This corresponds to taking f to be the identity in the definition of semitoric isomorphisms in Definition 1.3. Thus we see that if \sim represents the equivalence induced by semitoric isomorphisms we have that $\mathcal{T}_{\mathbb{T}} / \sim \subset \mathcal{T}_0$ so the metric on $\mathcal{T}_{\mathbb{T}}$ does produce a topology on a subset of \mathcal{T}_0 via the quotient topology.

In \mathcal{T}_0 we know that the semitoric invariant is a unique polygon so to conclude that the metrics produce the same topology it is sufficient to show that the same sequences of convex compact polygons converge with respect to both the Lebesgue measure and any admissible measure.

Lemma 2.14. *For $k \in \mathbb{N}$ let $\Delta_k, \Delta \subset \mathbb{R}^2$ be convex compact sets, let μ denote the Lebesgue measure on \mathbb{R}^2 and let ν be any admissible measure. Then $\lim_{k \rightarrow \infty} \mu(\Delta * \Delta_k) = 0$ if and only if $\lim_{k \rightarrow \infty} \nu(\Delta * \Delta_k) = 0$.*

Proof. If $\lim_{k \rightarrow \infty} \mu(\Delta * \Delta_k) = 0$ we can see that $\lim_{k \rightarrow \infty} \nu_0(\Delta * \Delta_k) = 0$ where ν_0 is the example of an admissible measure from Section 2.2. This is because $\nu_0(A) < \mu(A)$ for any set $A \subset \mathbb{R}^2$. Thus we conclude that $\lim_{k \rightarrow \infty} \nu(\Delta * \Delta_k) = 0$ by Lemma 2.8.

Now we will show the other direction. Suppose $\lim_{k \rightarrow \infty} \nu(\Delta * \Delta_k) = 0$ and fix $\varepsilon > 0$. Choose some $L > 0$ such that $\pi_1(\Delta) \subset [-L, L]$. By Lemma 2.7 we know there exists $x_0, y_0, y_1 \in \mathbb{R}$ with $y_0 < y_1$ and $x_0 \in [-L, L]$ such that the set $A = \{(x_0, y) \in \mathbb{R}^2 \mid y \in [y_0, y_1]\} \subset \Delta$ is a subset of Δ_k for $k > K_1$ for some fixed $K_1 \in \mathbb{N}$. Now, suppose that $k > K_1$ and $p \in \Delta_k$ has $\pi_1(p) > L + 1$. Then, since Δ_k is convex, the triangle with vertices $(x_0, y_0), (x_0, y_1), p$, which we will denote by G_p , must be a subset of Δ_k . Since $\pi_1(\Delta) \subset [-L, L]$ we know that $G_p \setminus \pi_1^{-1}([-L, L]) \subset \Delta * \Delta_k$ and the ν -measure of any such triangle G_p defined by a point $p \in \mathbb{R}^2$ with $\pi_1(p) > L$ is bounded below by a constant $c_1 = \nu(G_{p_0}) > 0$ where $p_0 = (L + 1, 0)$. This is because any triangle G_p where $\pi_1(p) > L$ contains a triangle $G_{(L+1, y)}$ for some $y \in \mathbb{R}$ and any such triangle is the image under a vertical, and thus ν -preserving, transformation of G_{p_0} . Similarly, $p \in \Delta_k$ for $k > K_1$ with $\pi_1(p) < -L$ would imply that $\nu(\Delta * \Delta_k) > c_2$ for some constant $c_2 > 0$. Thus, since $\lim_{k \rightarrow \infty} \nu(\Delta * \Delta_k) = 0$ we conclude that there exists some $K_2 > K_1$ such that $k > K_2$ implies that $\Delta_k \subset \pi_1^{-1}([-L, L])$. Since ν is admissible we know that there exists some $c_3 > 0$ such that $d\mu/d\nu < c_3$ on $\pi_1^{-1}([-L, L])$. Choose $K_3 > K_2$ such that $k > K_3$ implies that $\nu(\Delta * \Delta_k) < \varepsilon/c_3$ and notice that

$$\begin{aligned} \mu(\Delta * \Delta_k) &= \frac{d\mu}{d\nu} \nu(\Delta * \Delta_k) \\ &< c_3 \nu(\Delta * \Delta_k) < \varepsilon, \end{aligned}$$

because while the Radon-Nikodym derivative is not bounded on all of \mathbb{R}^2 it is bounded on the set $\Delta * \Delta_k$ for large enough k . \square

Corollary 2.15. *The metric d induces the same topology on $\mathcal{T}_{\mathbb{T}}$ as the metric defined in [27] does.*

To prove this result we had to assume that the polygons involved were compact. Of course, if we consider non-compact sets these metrics will not induce the same topology.

Remark 2.16. Let $\mathcal{T}^{\text{cpt}} \subset \mathcal{T}$ be the collection of *compact* semitoric integrable systems. Then the polygons produced will always be compact and thus Lemma 2.14 applies. So we can conclude that when restricting to \mathcal{T}^{cpt} the standard Lebesgue measure can be used in place of the choice of admissible measure and the same topology will be produced. \otimes

2.6. d and d^{Id} induce the same topology. Let

$$\mathcal{M}_{m_f, \vec{k}} = \{m \in \mathcal{M}_{m_f, [\vec{k}]} \mid m \text{ has twisting index } \vec{k}\}$$

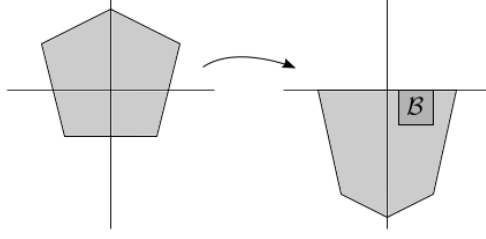


FIGURE 6. Without changing the ν -measure we can produce a new polygon which has $\{y = 0\}$ as its top boundary.

and define $d^{\text{Id}, \nu, \{b_n\}_{n=0}^\infty}$ on all of \mathcal{M} by

$$d^{\text{Id}, \nu, \{b_n\}_{n=0}^\infty}(m, m') = \begin{cases} d_{m_f, \vec{k}}^{\text{Id}, \nu, \{b_n\}_{n=0}^\infty}(m, m') & \text{if } m, m' \in \mathcal{M}_{m_f, \vec{k}} \text{ for some } m_f \in \mathbb{Z}_{\geq 0}, \vec{k} \in \mathbb{Z}^{m_f} \\ 1 & \text{otherwise.} \end{cases}$$

Both d and d^{Id} are defined on all of \mathcal{M} and the main result of this section will be that both of these metrics induce the same topology on \mathcal{M} .

Lemma 2.17. *Let $m, m_n \in \mathcal{M}$ for $n \in \mathbb{N}$. Then $d_{m_f, [\vec{k}]}^{\nu, \{b_n\}_{n=0}^\infty}(m, m_n) \rightarrow 0$ as $n \rightarrow \infty$ implies that $\lambda_j^n \rightarrow \lambda_j$ as $n \rightarrow \infty$ for all $j = 1, \dots, m_f$.*

Proof. Again we use d to denote $d_{m_f, [\vec{k}]}^{\nu, \{b_n\}_{n=0}^\infty}$ and d^p to denote $d_{m_f, \vec{k}}^{p, \nu, \{b_n\}_{n=0}^\infty}$.

Step 1: Let $p_n \in \mathcal{S}^{m_f}$ satisfy $d(m, m_n) = d^{p_n}(m, m_n)$ for each $n \in \mathbb{N}$. For the first step of this proof we will argue that $\lambda_{p_n(j)}^n \rightarrow \lambda_j$ by contrapositive. Suppose there exists some $j \in 1, \dots, m_f$ such that $\lambda_{p_n(j)}^n \not\rightarrow \lambda_j$. This means there exists $a > 0$ and a subsequence $(n_i)_{i=0}^\infty$ such that

$$|\lambda_{p_{n_i}(j)}^{n_i} - \lambda_j| > a \text{ for all } i \in \mathbb{N}.$$

Now let $t_j = t_{\ell_{\lambda_j}}^1$ and $t_j^{n_i} = t_{\ell_{\lambda_{p_{n_i}(j)}^{n_i}}}^1$. Let Δ be a polygon which represents a choice of $\vec{\varepsilon} = \{+1, \dots, +1\}$ for m . We must show that $\nu(t_j(\Delta) * t_j^{n_i}(\Delta))$ is bounded away from zero. We may assume that a is less than the horizontal distance from λ_j to the edge of the polygon Δ because $\min_{s \in \Delta} \pi_1(s) < \lambda_j < \max_{s \in \Delta} \pi_1(s)$. Let $b = \min_{x \in [\lambda_j - a, \lambda_j + a]} (\text{length}(\Delta \cap \ell_x))$ and notice that since Δ is a convex polygon we must have that $b > 0$.

The set Δ may be shifted by a vertical transformation so that $\max\{\pi_2(\Delta \cap \ell_x)\} = 0$ for each $x \in \mathbb{R}$ to form a new set $\Delta' \subset \mathbb{R}^2$. Let $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the composition of these transformations so $A(\Delta) = \Delta'$. This new set does not need to be convex but since ν is invariant under vertical translations we have that $\nu(\Delta) = \nu(\Delta')$. Notice that $\mathcal{B} = [\lambda_j - a, \lambda_j + a] \times [-b, 0]$ satisfies $\mathcal{B} \subset \Delta'$. See Figure 6.

Now there are two cases. If $\lambda_j < \lambda_j^{n_i}$ then we can see that $t_j(\mathcal{B}) \cap \{y > 0\} \subset t_j(\Delta') * t_j^{n_i}(\Delta')$. This is because $t_j^{n_i}$ is the identity on points where $x \in [\lambda_j - a, \lambda_j + a]$ and so for x in this interval Δ' does not intersect the open upper half plane. The set $t_j(\mathcal{B}) \cap \{y > 0\}$ always contains the rectangle $[\lambda_j + a/2, \lambda_j + a] \times [0, a/2]$. See Figure 7. Let $c_1 = \nu([\lambda_j + a/2, \lambda_j + a] \times [0, a/2])$.

Now suppose that $\lambda_j > \lambda_j^{n_i}$. In this case the symmetric difference always contains the region $t_j([\lambda_j, \lambda_j + a] \times [a - b, a])$ which has the same measure as $[\lambda_j, \lambda_j + a] \times [a - b, a]$, see

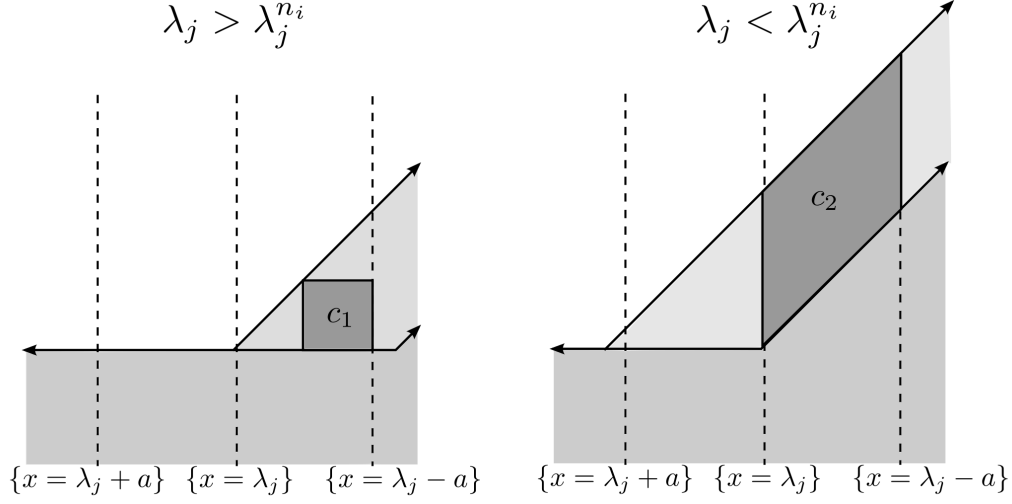


FIGURE 7. Either $\lambda_j^{n_i} < \lambda_j - a$ or $\lambda_j^{n_i} > \lambda_j + a$. Each case is shown above and in either case there is some positive measure region which is always in the symmetric difference. This causes convergence to be impossible.

Figure 7. Let $c_2 = \nu([\lambda_j, \lambda_j + a] \times [a - b, a])$ and let $c = \min\{c_1, c_2\}$. So in any case we have that $\nu(t_j(\Delta) * t_j^{n_i}(\Delta)) \geq c > 0$.

Notice that $\lim_{n \rightarrow \infty} d(m, m_n) = 0$ implies $\lim_{i \rightarrow \infty} \nu(\Delta * \Delta^{n_i}) = 0$. Fix $\varepsilon > 0$, assume that $\varepsilon < c$, and find $I > 0$ such that $i > I$ implies that $\nu(\Delta * \Delta^{n_i}) < \varepsilon$. Then for $i > I$ we have that

$$\begin{aligned} \nu[t_j(\Delta) * t_j^{n_i}(\Delta)] &\leq \nu[t_j(\Delta) * t_j^{n_i}(\Delta^{n_i})] + \nu[t_j^{n_i}(\Delta^{n_i}) * t_j^{n_i}(\Delta)] \\ &= \nu[t_j(\Delta) * t_j^{n_i}(\Delta^{n_i})] + \nu[\Delta^{n_i} * \Delta]. \end{aligned}$$

From this we conclude that

$$\begin{aligned} \nu[t_j(\Delta) * t_j^{n_i}(\Delta^{n_i})] &\geq \nu[t_j(\Delta) * t_j^{n_i}(\Delta)] - \nu[\Delta^{n_i} * \Delta] \\ &> c - \varepsilon. \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} \nu[t_j(\Delta) * t_j^n(\Delta^n)] = 0$ is impossible, but this is a term in $d(m, m_n)$.

Since there is a subsequence bounded away from zero we have that $d(m, m_n) \rightarrow 0$ is impossible so we conclude that $\lambda_{p_n(j)}^n \rightarrow \lambda_j$ for all $j = 1, \dots, m_f$.

Step 2: From Step 1 we know that $\lambda_{p_n(j)}^n \rightarrow \lambda_j$ as $n \rightarrow \infty$ for each $j = 1, \dots, m_f$. Let $D = \min\{|\lambda_j - \lambda_{j'}| \mid j, j' \in \{1, \dots, m_f\}, j \neq j'\}$. Then there exists some $N > 0$ such that $n > N$ implies that $|\lambda_{p_n(j)}^n - \lambda_j| < d/2$. Thus, for $n > N$ we have that $p_n = \text{Id}$ and the result follows. \square

Proposition 2.18. Let $m_f \in \mathbb{Z}_{\geq 0}$, $\vec{k} \in \mathbb{Z}^{m_f}$, $\{b_n\}_{n=0}^\infty$ be a linear summable sequence, and ν be an admissible measure. Then $d_{m_f, [\vec{k}]}^{\nu, \{b_n\}_{n=0}^\infty}$ and $d_{m_f, \vec{k}}^{\text{Id}, \nu, \{b_n\}_{n=0}^\infty}$ induce the same topology on \mathcal{M} .

Proof. Any sequence which converges for d^{Id} will converge for d because $d < d^{\text{Id}}$. Suppose that $(m_n)_{n=1}^\infty$ is a sequence in \mathcal{M} which converges to $m \in \mathcal{M}$ with respect to d . Then by Step 2 of the proof of Lemma 2.17 we know there exists some $N > 0$ such that for $n > N$ we have

that $d(m, m_n) = d^{\text{ld}}(\tau, \tau_n)$. Thus we see that the sequence $d^{\text{ld}}(m, m_n)$ is eventually equal to a sequence which converges to zero, so we conclude that $d^{\text{ld}}(m, m_n) \rightarrow 0$ as $n \rightarrow \infty$. \square

Remark 2.19. Each $\mathcal{M}_{m_f, \vec{k}} \subset \mathcal{M}$ is in a separate component of (\mathcal{M}, d) . This is because these are defined to be in different components for d^{ld} and we have just shown that d^{ld} and d induce the same topology. \oslash

3. THE COMPLETION

In this section we compute the completion of the space of semitoric ingredients \mathcal{M} which corresponds to the completion of \mathcal{T} by Theorem 1.17. We know that the completion of an open interval in \mathbb{R} with the usual metric is the corresponding closed interval and we have already stated that $\mathbb{R}[[X, Y]]_0$ is complete (Proposition 1.20), so to produce the completion of \mathcal{M} it seems the only difficulty will be with the weighted polygons. This is not the case since in fact defining the distance as a minimum of permutations has intertwined the metrics on these different spaces so we can not consider them separately. This section has similar arguments to those in [27] except that in our case we must consider a whole family of polygons all at once instead of only one polygon. For the remainder of this section fix some admissible measure ν , some linear summable sequence $\{b_n\}_{n=0}^\infty$, a nonnegative integer m_f , and a vector $\vec{k} \in \mathbb{Z}^{m_f}$. For simplicity we will use d and d^p to refer to $d_{m_f, [\vec{k}]}^{\nu, \{b_n\}_{n=0}^\infty}$ and $d_{m_f, \vec{k}}^{p, \nu, \{b_n\}_{n=0}^\infty}$ respectively (for $p \in \mathcal{S}^{m_f}$).

In Section 3.1 we show that the completion must contain $\tilde{\mathcal{M}}$ and in the remaining subsections we show that $\tilde{\mathcal{M}}$ is complete. We Section 3.2 we prove several Lemmas about Cauchy sequences which are used in Section 3.3 to conclude that $\tilde{\mathcal{M}}$ is in fact the completion of \mathcal{M} .

There is no way for elements of \mathcal{M} with different numbers of focus-focus points or twisting indices that are not equivalent (under the equivalence from 1.23) can be close to one another because the distance between any two such systems is always 1 (see Definition 1.29). Thus, we will work with the components $\mathcal{M}_{m_f, [\vec{k}]}$ of \mathcal{M} .

Now we must notice that the definition of d from Definition 1.28 holds on $\tilde{\mathcal{M}}$ as well. That is, extend the definition of d in the following way:

Definition 3.1. Let $\{b_n\}_{n=0}^\infty$ be a linear summable sequence, ν be an admissible measure, $m_f \in \mathbb{Z}_{\geq 0}$, and $\vec{k} \in \mathbb{Z}^{m_f}$. Suppose that $m, m' \in \tilde{\mathcal{M}}$ denote $([A_w], (h_j)_{j=1}^{m_f}, ((S_j)^\infty)_{j=1}^{m_f})$ and $([A'_w], (h'_j)_{j=1}^{m_f}, ((S'_j)^\infty)_{j=1}^{m_f})$ respectively. Then we define:

(1) the *comparison with alignment* p to be

$$d_{m_f, \vec{k}}^{p, \nu, \{b_n\}_{n=0}^\infty}(m, m') = d_{\mathcal{P}}^{p, \nu}([A_w], [A'_w]) + \sum_{j=1}^{m_f} \left(d_0^{\{b_n\}_{n=0}^\infty}((S_j)^\infty, (S_{p(j)})^\infty) + d_h(h_j, h'_{p(j)}) \right);$$

(2) the *distance between* m and m' to be

$$d_{m_f, [\vec{k}]}^{\nu, \{b_n\}_{n=0}^\infty}(m, m') = \min_{p \in \mathcal{S}_{\vec{k}, \vec{k}'}^{m_f}} \{d_{m_f, \vec{k}}^{p, \nu, \{b_n\}_{n=0}^\infty}(m, m')\}.$$

Proposition 3.2. d is a metric on $\tilde{\mathcal{M}}$.

This proposition follows from the proof of Proposition 2.13.

Remark 3.3. Notice that d^{Id} is not a metric on $\tilde{\mathcal{M}}$ because it does not satisfy the triangle inequality. This can be seen in the example explained in Remark 3.16. \odot

Throughout Section 3.1 each space we examine can be viewed as a subspace of $\tilde{\mathcal{M}}$ and we will endow them with the structure of a metric subspace.

Remark 3.4. The space \mathcal{M} can be viewed as a subspace of $\tilde{\mathcal{M}}$ because there is a natural correspondence between the elements of \mathcal{M} and the elements of a subset of $\tilde{\mathcal{M}}$. This is because there is at most one element of \mathcal{M} in each equivalence class in $\tilde{\mathcal{M}}$ so the space \mathcal{M} corresponds to the subset $\{[m] \mid m \in \mathcal{M}\}$. \odot

3.1. The completion must contain $\widetilde{\mathcal{M}_{m_f, [\vec{k}]}}$. In the next few lemmas we start with $\mathcal{M}_{m_f, [\vec{k}]}$ and build up to $\widetilde{\mathcal{M}_{m_f, [\vec{k}]}}$ in several steps, showing that each inclusion is dense. First we will show that the completion of $\mathcal{M}_{m_f, [\vec{k}]}$ must include at least all rational labeled polygons which satisfy the convexity requirements.

Lemma 3.5. Let $\mathcal{P}'_{m_f, [\vec{k}]} \subset \widetilde{\mathcal{DPoly}_{m_f, [\vec{k}]}}(\mathbb{R}^2)$ be given by

$$\mathcal{P}'_{m_f, [\vec{k}]} = \left\{ [(\Delta, (\ell_{\lambda_j}, +1, k'_j)_{j=1}^{m_f})] \left| \begin{array}{l} t_{\vec{u}}(\Delta) \in \text{Polyg}(\mathbb{R}^2) \text{ for any } \vec{u} \in \{0, 1\}^{m_f}, \\ \vec{k} \sim \vec{k}', \nu(\Delta) < \infty, \\ \min_{s \in \Delta} \pi_1(s) < \lambda_1 < \dots < \lambda_{m_f} < \min_{s \in \Delta} \pi_1(s) \end{array} \right. \right\}$$

and let

$$\mathcal{M}'_{m_f, [\vec{k}]} = \mathcal{P}'_{m_f, [\vec{k}]} \times [0, 1]^{m_f} \times \mathbb{R}[[X, Y]]_0^{m_f}.$$

Then the inclusion $\mathcal{M}_{m_f, [\vec{k}]} \subset \mathcal{M}'_{m_f, [\vec{k}]}$ is dense.

Proof. Fix any element $m = ([\Delta_w], (h_j)_{j=1}^{m_f}, ((S_j)^\infty)_{j=1}^{m_f}) \in \mathcal{M}'_{m_f, [\vec{k}]}$. Since $d \leq d^{\text{Id}}$ we will show there exists an element $m' \in \mathcal{M}_{m_f, [\vec{k}]}$ arbitrarily close to m with respect to the function d^{Id} . Clearly we will have no problems with making the volume invariant or the Taylor series arbitrarily close so just consider the polygons.

Let $[\Delta_w] = [(\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})] \in \mathcal{P}'_{m_f, [\vec{k}]}$ and fix $\varepsilon > 0$. We will show there exists some element $[\Delta'_w] \in \mathcal{DPolyg}_{m_f, [\vec{k}]}(\mathbb{R}^2)$ such that $d_P^{\text{Id}, \nu}([\Delta_w], [\Delta'_w]) < \varepsilon$. We will choose this element of $\mathcal{DPolyg}_{m_f, [\vec{k}]}(\mathbb{R}^2)$ to have the same λ_j values as Δ_w . Since the action of $t_{\vec{u}}, \vec{u} \in \{0, 1\}^{m_f}$, does not change the volume of sets this means that

$$d_P^{\text{Id}}([\Delta_w], [\Delta'_w]) \leq 2^{m_f} \nu(\Delta * \Delta')$$

where $(\Delta', (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f}) \in [\Delta'_w]$. This means to complete the proof it suffices to show that there exists an element $(\Delta', (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f}) \in \mathcal{DPolyg}_{m_f, [\vec{k}]}(\mathbb{R}^2)$ such that Δ and Δ' are equal except on a set of ν -measure less than $2^{-m_f} \varepsilon$.

For $j = 1, \dots, m_f$ let $p_j \in \mathbb{R}^2$ be the intersection of ℓ_{λ_j} with the top boundary of Δ . Let $U \subset \mathbb{R}^2$ be a union of disjoint neighborhoods around each corner of Δ which is not an element of $\{p_j\}_{j=1}^{m_f}$ such that $\nu(U) < \varepsilon/2^{m_f+1}$. Also, let $V \subset \mathbb{R}^2 \setminus U$ be a union of disjoint neighborhoods around each point p_j for each $j = 1, \dots, m_f$ and $\nu(V) < \varepsilon/2^{m_f+1}$. We will define Δ' in several

stages, editing it several times. Start by assuming that $\Delta' = \Delta$. By [27, Remark 23] we can edit Δ' on the set U so that every vertex is Delzant except possibly the ones in V .

Now, recall that for a *semitoric* polygon to be Delzant the points p_j must all either be fake or hidden Delzant corners. This is equivalent to saying that the corners on the top boundary of $t_{\vec{u}}(\Delta')$ must all be Delzant for $\vec{u} = \langle 1, \dots, 1 \rangle$. Since $t_{\vec{u}}(\Delta')$ is a convex polygon and $t_{\vec{u}}(V)$ is a neighborhood of the edges $t_{\vec{u}}(p_j)$ we can again use [27, Remark 23] to conclude that we may edit $t_{\vec{u}}(\Delta')$ inside of the set V such that all of the vertices on the top boundary are Delzant. Now we have finished defining $t_{\vec{u}}(\Delta')$ and since this map is invertible we have also defined Δ' . Notice for $j = 1, \dots, m_f$ each point $t_{\vec{u}}(p_j)$ is either a Delzant corner, which would make p_j a hidden Delzant corner, or it is not a vertex at all, in which case p_j would be a fake corner. Also, it is easy to check that any new Delzant corner we had to define in $t_{\vec{u}}(V)$ which is not on the point $t_{\vec{u}}(p_j)$ for some $j = 1, \dots, m_f$ gets transformed by $t_{\vec{u}}^{-1}$ to form a Delzant corner on Δ' . In conclusion, $[\Delta'_w]$ is a Delzant semitoric polygon and each of the 2^{m_f} polygons in the equivalence class is equal to each polygon in $[\Delta_w]$ except on a set of ν -measure less than $\varepsilon/2^{m_f}$. □

So from the above Lemma we conclude that the completion of $\mathcal{M}_{m_f, [\vec{k}]}$ must contain $\mathcal{M}'_{m_f, [\vec{k}]}$. In the next Lemma we show it must contain a larger set. The only difference between $\mathcal{P}'_{m_f, [\vec{k}]}$ and $\mathcal{P}''_{m_f, [\vec{k}]}$ is that $\mathcal{P}''_{m_f, [\vec{k}]}$ allows irrational polygons.

Lemma 3.6. *Let*

$$\mathcal{P}''_{m_f, [\vec{k}]} = \left\{ [(\Delta, (\ell_{\lambda_j}, +1, k'_j)_{j=1}^{m_f})] \left| \begin{array}{l} t_{\vec{u}}(\Delta) \text{ is a convex polygon for any } \vec{u} \in \{0, 1\}^{m_f}, \\ 0 < \nu(\Delta) < \infty, \vec{k} \sim \vec{k}' \\ \min_{s \in \Delta} \pi_1(s) < \lambda_1 < \dots < \lambda_{m_f} < \max_{s \in \Delta} \pi_1(s) \end{array} \right. \right\}$$

and let

$$\mathcal{M}''_{m_f, [\vec{k}]} = \mathcal{P}''_{m_f, [\vec{k}]} \times [0, 1]^{m_f} \times \mathbb{R}[[X, Y]]_0^{m_f}.$$

Then the inclusion $\mathcal{M}'_{m_f, [\vec{k}]} \subset \mathcal{M}''_{m_f, [\vec{k}]}$ is dense.

Proof. Just as in the proof of Lemma 3.5 we can see that we only need to consider the polygons. Suppose that $[\Delta_w] \in \mathcal{P}''_{m_f, [\vec{k}]}$ and $(\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f}) \in [\Delta_w]$. Given any $\varepsilon > 0$ we can find an open neighborhood of the boundary of Δ which has ν -measure less than ε (since the boundary has measure zero and ν is regular) and we may approximate Δ by a rational polygon with boundary inside of this neighborhood. In the case that Δ is compact this can be done by approximating the irrational slopes with rational ones (exactly as done in [27]).

This strategy will work even if Δ is not compact. For the faces of Δ which are non-compact with irrational slope (if there are any) we can still approximate these with a line of rational slope because of the properties of the admissible measure ν . Suppose there is a non-compact face of Δ which has irrational slope $r \in \mathbb{R} \setminus \mathbb{Q}$. Then choose $q \in \mathbb{Q}$ such that $q < r$ and $\nu(\{qx < y < rx\}) < \varepsilon$ and let the edge on the rational polygon have slope q . Such a slope can be chosen because if the measure of that set is always finite and if is too large then replacing q by $q_2 = q + r/2$ will produce a wedge with half the measure of the original. □

For the next Lemma we only slightly change the restrictions on the $(\lambda_j)_{j=1}^{m_f}$.

Lemma 3.7. *Let*

$$\mathcal{P}'''_{m_f, [\vec{k}]} = \left\{ [(\Delta, (\ell_{\lambda_j}, +1, k'_j)_{j=1}^{m_f})] \left| \begin{array}{l} t_{\vec{u}}(\Delta) \text{ is a convex polygon for any } \vec{u} \in \{0, 1\}^{m_f}, \\ 0 < \nu(\Delta) < \infty, \vec{k} \sim \vec{k}', \\ \lambda_j \in \mathbb{R} \cup \{\infty\} \text{ for } j = 1, \dots, m_f, \text{ and} \\ \min_{s \in \Delta} \pi_1(s) \leq \lambda_1 \leq \dots \leq \lambda_{m_f} \leq \max_{s \in \Delta} \pi_1(s) \end{array} \right. \right\}$$

and let

$$\mathcal{M}'''_{m_f, [\vec{k}]} = \mathcal{P}'''_{m_f, [\vec{k}]} \times [0, 1]^{m_f} \times \mathbb{R}[[X, Y]]_0^{m_f}.$$

The inclusion $\mathcal{M}''_{m_f, [\vec{k}]} \subset \mathcal{M}'''_{m_f, [\vec{k}]}$ is dense.

Remark 3.8. Since it is possible for $\lambda_j = \lambda_{j+1}$ for some $j \in 1, \dots, m_f - 1$ the order that the critical points are labeled in a system cannot be made unique by only considering that λ values. This means there could be two elements in $\widetilde{\mathcal{M}_{m_f, [\vec{k}]}}$ which have the same invariants except labeled in a different order. Of course we do not want this because these two elements should be the same, so we use the other invariants to create a unique ordering on the critical points of any element of $\widetilde{\mathcal{M}_{m_f, [\vec{k}]}}$. We fix the order so that if $\lambda_j = \lambda_{j+1}$ for some $j = 1, \dots, m_f - 1$ then we require that $h_j \leq h_{j+1}$. In the case that $\lambda_j = \lambda_{j+1}$ and $h_j = h_{j+1}$ we look to the Taylor series. In this situation we require that the coefficient of X of the Taylor series $(S_j)^\infty$ is less than or equal to the coefficient of X in $(S_{j+1})^\infty$ and if those are equal we look to the coefficient of Y and continue in this fashion. Now given any system with critical points there is a unique order in which to label them which is essentially the lexicographic order on the invariants. \oslash

Notice that we allow (positive only) infinite values for the λ_j . This can only happen in the case that the polygon is non-compact. If $\lambda_j = +\infty$ then we define t_j^1 to be the identity because all of \mathbb{R}^2 is to the left of this value.

Proof of Lemma 3.7. Again, we only need to consider the polygons. We will prove this Lemma in two steps. First, suppose that $[\Delta_w] \in \mathcal{P}'''_{m_f, [\vec{k}]}$ has $\lambda_j < \infty$ for each $j = 1, \dots, m_f$ so the only thing that is keeping $[\Delta_w]$ from being in $\mathcal{P}''_{m_f, [\vec{k}]}$ is the possibility that $\lambda_j = \lambda_{j+1}$ for some fixed $j \in \{1, \dots, m_f - 1\}$. Let \vec{u} be all zeros except for a 1 in the j^{th} and $(j+1)^{st}$ positions. Then $[\Delta_w] \in \mathcal{P}'''_{m_f, [\vec{k}]}$ implies that $t_{\vec{u}}(\Delta)$ is convex so we know that there is a vertex of Δ on the top boundary with x -coordinate λ_j . Let m_1 denote the slope of the edge to the left of this vertex and let m_2 denote the slope to the right. Then we can see that the convexity of $t_{\vec{u}}(\Delta)$ implies that $m_1 \geq m_2 + 2$. Now we want to show that there exists some $[\Delta'_w] \in \mathcal{P}''_{m_f, [\vec{k}]}$ arbitrarily close in d^{Id} to $[\Delta_w]$. Let $[\Delta'_w]$ be equal to $[\Delta_w]$ except that $\lambda'_j < \lambda_j < \lambda'_{j+1}$ and that the top boundary of Δ' has slope $m_1 - 1$ on the interval $x \in (\lambda'_j, \lambda'_{j+1})$. So, as is shown in Figure 8, we have cut the corner off of Δ to produce Δ' and clearly this cut can be made as small as desired. This process can be repeated for each instance of $\lambda_j = \lambda_{j+1}$ for $j \in \{1, \dots, m_f\}$.

Now we proceed to step two. Assume that $[\Delta_w] = [(\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})] \in \widetilde{\mathcal{DPolyg}_{m_f, [\vec{k}]}}(\mathbb{R}^2)$ has $\lambda_{m_f} = +\infty$ (and $\lambda_j < \infty$ for $j = 1, \dots, m_f - 1$) and we will construct a sequence with

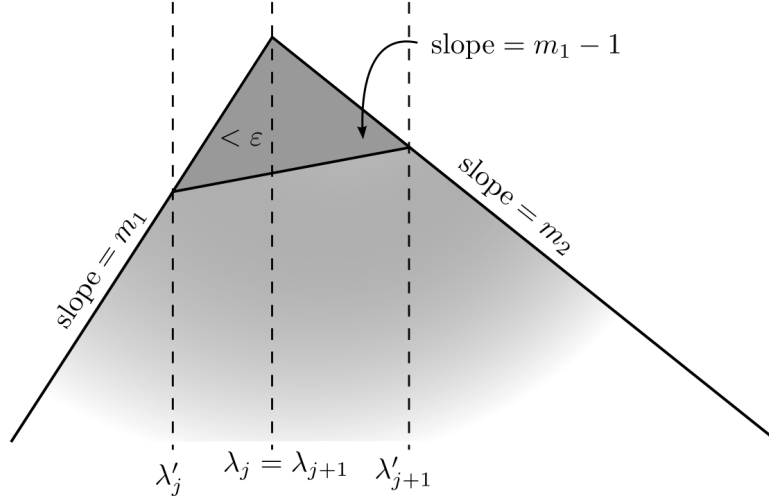


FIGURE 8. By cutting the corner and adjusting the values of λ_j and λ_{j+1} of an element in $\mathcal{P}'''_{m_f, [\vec{k}]}$ we can produce an element of $\mathcal{P}''_{m_f, [\vec{k}]}$ which is very close.

$[\Delta_w]$ as its limit. Let $N = \max_{j=1, \dots, m_f-1} |\lambda_j|$ and for any $n \in \mathbb{N}$ which satisfies $n > N$ define a set $\Delta^n = \Delta \cap [-n, n]$ with $\lambda_{m_f} = n$. That is

$$[\Delta_w^n] = (\Delta^n, (\ell_{\lambda_j}, +1)_{j=1}^{m_f-1}, (\ell_n, +1)).$$

Notice that each polygon in each family $[\Delta_w^n]$ is convex because it is the intersection of two convex sets. Then $d_P^\nu([\Delta_w], [\Delta_w^n]) \rightarrow 0$. Clearly a similar process can be used to produce sets which have multiple λ values which are infinite.

□

Next we would like to consider arbitrary convex sets, but there is a subtlety. We must instead consider all sets which are convex up to measure zero corrections (as is done in [27]). So far we have only been working with polygons and if the symmetric difference of two polygons has zero measure in ν (and therefore also in μ) we know that those polygons are actually the same set. Of course, this is not true for arbitrary subsets of \mathbb{R}^2 . Recall that ν and the Lebesgue measure μ have exactly the same measure zero sets, so the equivalence relation in the following definition does not depend on the choice of admissible measure.

Definition 3.9. Let

$$\mathcal{C}_{m_f, [\vec{k}]} = \left\{ [(A, (\ell_{\lambda_j}, +1, k'_j)_{j=1}^{m_f})] \left| \begin{array}{l} A \subset \mathbb{R}^2, \lambda_j \in \mathbb{R} \cup \{\infty\} \text{ for } j = 1, \dots, m_f, \\ t_{\vec{u}}(A) \text{ is a convex set for any } \vec{u} \in \{0, 1\}^{m_f}, \\ \vec{k} \sim \vec{k}', 0 < \nu(\Delta) < \infty, \text{ and} \\ \min_{s \in A} \pi_1(s) \leq \lambda_1 \leq \dots \leq \lambda_{m_f} \leq \max_{s \in A} \pi_1(s) \end{array} \right. \right\}.$$

Further, for any measurable sets $A, B \subset \mathbb{R}^2$ we say $A \simeq B$ if and only if $\mu(A * B) = 0$ and let $[A]$ denote the equivalence class of A with respect to this relation. Finally, let

$$\widetilde{\mathcal{DPolyg}_{m_f, [\vec{k}]}(\mathbb{R}^2)} = \left\{ [([A], (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})] \left| \begin{array}{l} [(A, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})] \in \mathcal{C}_{m_f, [\vec{k}]} \text{ or} \\ \nu(A) = 0 \text{ and } \lambda_j = 0 \text{ for } j = 1, \dots, m_f \end{array} \right. \right\}^{20}.$$

For the last Lemma in this section we will show that the inclusion in $\widetilde{\mathcal{M}_{m_f, [\vec{k}]}}$, which is defined in Definition 1.30, is also dense.

Lemma 3.10. *The inclusion $\mathcal{M}_{m_f, [\vec{k}]}''' \subset \widetilde{\mathcal{M}_{m_f, [\vec{k}]}}$ is dense²¹.*

Proof. Once more, we only must consider the labeled weighted convex sets since it is easy to align the volume invariant and Taylor series invariant. Let $[(A, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})] = [\Delta_w] \in \widetilde{\mathcal{DPolyg}_{m_f, [\vec{k}]}(\mathbb{R}^2)}$. Now pick $[(B, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})] \in \mathcal{P}_{m_f, [\vec{k}]}'''$ and notice that they have the same λ values, so if A and B are close then so are all of the other polygons. Simply approximate A by a family of disjoint rectangles contained in A . We need to be sure that $t_{\vec{u}}(B)$ is convex for any choice of $\vec{u} \in \{0, 1\}^{m_f}$ so take B to be the convex hull of the rectangles which approximate A from the inside and the points in the top boundary of A which have x -value equal to λ_j for some $j \in \{1, \dots, m_f\}$. Since $B \subset A$ and $t_{\vec{u}}(A)$ is convex around $x = \lambda_j$ for each $j = 1, \dots, m_f$ we know that $t_{\vec{u}}(B)$ is convex (Figure 9). \square

So now from the results of Lemma 3.5, Lemma 3.6, Lemma 3.7, and Lemma 3.10 the following lemma is immediate.

Lemma 3.11. *The completion of $\mathcal{M}_{m_f, [\vec{k}]}$ must contain $\widetilde{\mathcal{M}_{m_f, [\vec{k}]}}$.*

3.2. Cauchy sequences for d and d^{Id} .

Lemma 3.12. *Let $m_n \in \widetilde{\mathcal{M}_{m_f, [\vec{k}]}}$ for $n = 1, \dots, \infty$. If $(m_n)_{n=1}^{\infty}$ is Cauchy with respect to d then there exists a subsequence $(m_{n_i})_{i=1}^{\infty}$ which is Cauchy with respect to d^{Id} .*

Proof. Let $(m_n)_{n=1}^{\infty}$ be as in the statement of the Lemma. Let $A_0 = \mathbb{N}$ and let $M_0 = 0$. We will define A_n and M_n recursively for each $n \in \mathbb{N}$. Suppose that $|A_{n-1}| = \infty$ and $M_{n-1} \in A_{n-1}$. Let $\varepsilon_n = 2^{-n}$. Find some $M > 0$ such that $k, l > M$ implies that $d(m_k, m_l) > \varepsilon_n/2$. Now let M_n be any element of A_{n-1} which is greater than M and M_{n-1} . This means $d(m_{M_n}, m_l) < \varepsilon_n/2$ for any $l > M_n$. For $p \in \mathcal{S}^{m_f}$ let $\mathcal{B}_p^n = \{l \in A_{n-1} \mid l > M_n, d^p(m_{M_n}, m_l) <$

²⁰Here it is important to notice that we have included one extra element in each $\widetilde{\mathcal{DPolyg}_{m_f, [\vec{k}]}(\mathbb{R}^2)}$, the equivalence class of the empty set. For this element the values of λ_j are unimportant so we set them all equal to zero (in fact, any fixed number will do).

²¹The explanation of how $\mathcal{M}_{m_f, [\vec{k}]}''$ can be viewed as a subspace of $\widetilde{\mathcal{M}_{m_f, [\vec{k}]}}$ is in Remark 3.4.

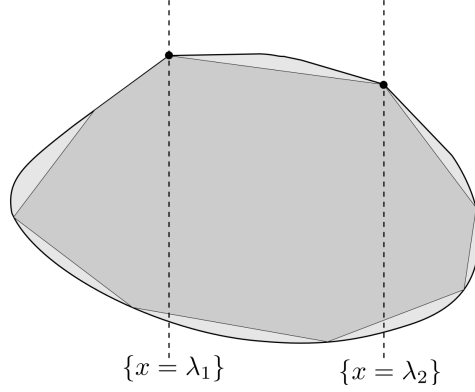


FIGURE 9. An arbitrary convex set can be approximated from the inside by a polygon. The convexity requirements will be met as long as the vertices on the top boundary at $\{x = \lambda_j\}$ for each $j = 1, \dots, m_f$ are included in the polygon.

$\varepsilon_n/2\}$. Notice that $\cap_{p \in \mathcal{S}^{m_f}} \mathcal{B}_p^n = \{l \in A_{n-1} \mid l > M_n\}$ by the definition of M_n . The union of this finite number of sets has infinite cardinality so at least one of those sets must also have infinite cardinality. Choose any $p_n \in \mathcal{S}^{m_f}$ such that $|\mathcal{B}_{p_n}^n| = \infty$ (there may be several possible choices). Now define $A_n = (A_{n-1} \cap [0, M_n]) \cup \mathcal{B}_{p_n}^n$. Notice that $M_n \in A_n$ and $|A_n| = \infty$.

Now let $A = \cap_{n \in \mathbb{N}} A_n$ and notice that $|A| = \infty$ because $\{M_n \mid n \in \mathbb{N}\} \subset A$.

So $(m_a)_{a \in A}$ is a subsequence of $(m_n)_{n=1}^\infty$. We will show that this subsequence is Cauchy with respect to d^{Id} . Fix any $\varepsilon > 0$ and find $n \in \mathbb{N}$ such that $\varepsilon_n < \varepsilon$. Now pick any $k, l > M_n$ with $k, l \in A$. Then $k, l \in A_n$ implies that $k, l \in \mathcal{S}_{p_n}^n$ so $d^{p_n}(m_{M_n}, m_k), d^{p_n}(m_{M_n}, m_l) < \varepsilon_n/2 < \varepsilon/2$. Also notice that p_n being an appropriate permutation to compare m_k with m_{M_n} and also appropriate to compare m_l with m_{M_n} implies that $\text{Id} \in \mathcal{S}^{m_f}$ is an appropriate permutation to compare m_k and m_l . Thus

$$d^{\text{Id}}(m_k, m_l) \leq d^{p_n}(m_k, m_{M_n}) + d^{p_n}(m_l, m_{M_n}) < \varepsilon$$

by Lemma 2.11. □

Lemma 3.13. *Suppose that $(m_n)_{n=1}^\infty$ is a sequence of elements of $\widetilde{\mathcal{M}_{m_f, [\vec{k}]}}$ which is Cauchy with respect to the function d^{Id} . Then there exists some $p \in \mathcal{S}^{m_f}$ and $m \in \widetilde{\mathcal{M}_{m_f, [\vec{k}]}}$ such that*

$$\lim_{n \rightarrow \infty} d^p(m_n, m) = 0.$$

Proof. For $A, B \subset \mathbb{R}^2$ say $A \simeq B$ if and only if $\nu(A * B) = 0$ and let \mathcal{F} denote the subsets of \mathbb{R}^2 with finite ν -measure modulo \simeq . Now let $\mathcal{E} = \{[A] \in \mathcal{F} \mid \text{there exists } B \in [A] \text{ which is convex}\}$ and let $d_{\mathcal{E}}$ be the metric on this space given by the ν -measure of the symmetric difference. For simplicity we will write $A \in \mathcal{E}$ instead of $[A] \in \mathcal{E}$. We will show that this metric space is complete. Let χ_A denote the characteristic function of the set $A \in \mathcal{E}$. Then for $A, B \in \mathcal{E}$ we can see that

$$d_{\mathcal{E}}(A, B) = \int_{\mathbb{R}^2} |\chi_A - \chi_B| d\nu = \|\chi_A - \chi_B\|_{L^1}$$

the L^1 norm on (\mathbb{R}^2, ν) . Now suppose that $(A^k)_{k=1}^\infty$ is a Cauchy sequence in $(\mathcal{E}, d_\mathcal{E})$ and by measure zero adjustments we can assume that each A^k is convex. Then $(\chi_{A^k})_{k=1}^\infty$ is Cauchy in $L^1(\mathbb{R}^2, \nu)$ and thus there must exist some function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined up to measure zero such that

$$\lim_{k \rightarrow \infty} \|g - \chi_{A^k}\|_{L^1} = 0$$

because L^1 is complete.

The functions $(\chi_{A^k})_{k=1}^\infty$ converge to g in L^1 so we know that there is a subsequence $(\chi_{A^{k_n}})_{n=1}^\infty$ which converges to g pointwise off of some measure zero set S . Let

$$A = \{x \in \mathbb{R}^2 / S \mid g(x) = 1\}$$

and now we will show that A is almost everywhere equal to a convex set so \mathcal{E} is complete. Let A' be the convex hull of A and we will show that $\nu(A * A') = 0$. Let $p \in A'$ which means there exists $q, r \in A$ and $t \in [0, 1]$ such that $p = (1-t)q + tr$. Since the subsequence $(\chi_{A^{k_n}})_{n=1}^\infty$ converges pointwise to χ_A at the points q and r (since $q, r \in A$ and A is disjoint from S) this means that there exists some $N > 0$ such that $n > N$ implies $q, r \in A^{k_n}$. Thus, since each A^k is convex we see that for $n > N$ we have $p \in A^{k_n}$. We conclude that $p \in A \cup S$ and thus $A * A' \subset S$ so $\nu(A * A') = 0$. Also notice $\nu(A^k, A) \rightarrow 0$ as $k \rightarrow \infty$ implies that $\nu(A) < \infty$. This means $A \in \mathcal{E}$ so $(\mathcal{E}, d_\mathcal{E})$ is a complete metric space.

Let $([A_w^k])_{k=1}^\infty$ be a Cauchy sequence in $(\widetilde{\mathcal{DPolyg}_{m_f, [\vec{k}]}}(\mathbb{R}^2), d_P^{\text{Id}, \nu})$. Let

$$[A_w^k] = [(A^k, (\ell_{\lambda_j^k}, +1, k_j)_{j=1}^{m_f})] \text{ and let } A_\epsilon^k = t_{\vec{u}}^k(A^k)$$

for each $\vec{\epsilon} \in \{-1, 1\}^{m_f}$ with $u_j = \frac{1-\epsilon_j}{2}$. Since this sequence is Cauchy we also know that the sequence $(A_\epsilon^k)_{k=1}^\infty$ is a Cauchy sequence in $(\mathcal{E}, d_\mathcal{E})$. Thus for each $\vec{\epsilon} \in \{-1, 1\}^{m_f}$ there exists some convex $A_\epsilon \in \mathcal{E}$ which is the limit of $(A_\epsilon^k)_{k=1}^\infty$ in \mathcal{E} . Let $A = A_{(1, \dots, 1)}$. We have produced a family of convex, ν -finite sets which could be the limit, but we still need to check that there is some choice of $(\Lambda_j)_{j=1}^{m_f}$ such that $A_\epsilon = t_{\vec{u}}(A_0)$ in \mathcal{E} for each $j = 1, \dots, m_f$.

Fix some $j \in \{1, \dots, m_f\}$ and let $A_j = A_\epsilon$ where $\epsilon_j = -1$ and $\epsilon_i = +1$ for $i \neq j$ and let t_k denote $t_{\lambda_j^k}^1$. Since ν is invariant under vertical translations we have that

$$d_\mathcal{E}(t_k(A), t_{\vec{u}}(A^k)) = d_\mathcal{E}(A, A^k)$$

so both go to zero as $k \rightarrow \infty$. By the triangle inequality we can see that

$$d_\mathcal{E}(t_k(A), A_j) \leq d_\mathcal{E}(t_k(A), t_k(A^k)) + d_\mathcal{E}(t_k(A^k), A_j)$$

so we conclude that

$$(3) \quad d_\mathcal{E}(t_k(A), A_j) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

If $(\lambda_j^k)_{k=1}^\infty$ diverges to $+\infty$ or converges to $\sup(\pi_1(A))$ then we are done. This is because in this case $d_\mathcal{E}(t^k(A), A_j) \rightarrow 0$ as $k \rightarrow \infty$ implies that A and A_j represent the same element in \mathcal{E} (i.e. they are equal almost everywhere) and t^Λ acts as the identity on A_0 if Λ is the rightmost value of A_0 .

Otherwise we can find some $x_0, a \in \mathbb{R}$ with $a > 0$ such that $[x_0, x_0 + 2a] \subset \pi_1(A)$ and there exists a subsequence $(\lambda_j^{k_n})_{n=1}^\infty$ such that $\lambda_j^{k_n} < x_0$ for all n . Notice that $A \cap \ell_x$ is an interval for any $x \in \pi_1(A)$ because A is convex. Let $\delta_1 = \text{length}(A \cap \ell_{x_0})$ and $\delta_2 = \text{length}(A \cap \ell_{x_0+a})$ and notice that $\delta_1, \delta_2 < \infty$ because otherwise we would have $\nu(A) = \infty$

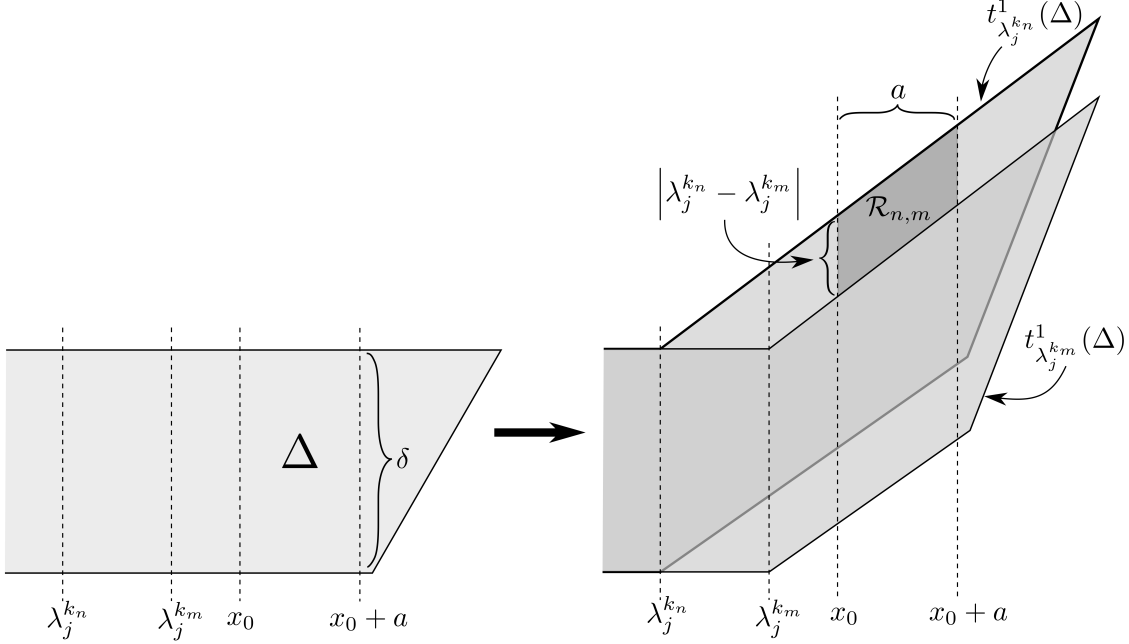


FIGURE 10. The action of $t_{\lambda_j^{k_m}}^1$ and $t_{\lambda_j^{k_n}}^1$ on a polygon. It can be seen that $\mathcal{R}_{n,m}$ is a subset of the symmetric difference and has measure which is nonzero if $|\lambda_j^{k_n} - \lambda_j^{k_m}| \neq 0$.

or $\nu(A) = 0$ because A is convex and ν is invariant under vertical translations. Also notice that $\text{length}(A \cap \ell_x) \geq \min\{\delta_1, \delta_2\}$ for any $x \in [x_0, x_0 + a]$ because A is convex. Pick any $n, m \in \mathbb{N}$ and we can see that $t_{\lambda_j^{k_n}}^1$ and $t_{\lambda_j^{k_m}}^1$ only differ by a vertical translation when acting on $A \cap \pi_1^{-1}([x_0, x_0 + a])$ (see Figure 10). This guarantees that there is a region $\mathcal{R}_{n,m}$ in the symmetric difference $t_{\lambda_j^{k_n}}^1(A) * t_{\lambda_j^{k_m}}^1(A)$ which has the same measure as a rectangle of length a and height $\min\{\delta_1, \delta_2, |\lambda_j^{k_n} - \lambda_j^{k_m}|\}$ positioned between the x -values of x_0 and $x_0 + a$ (since ν is translation invariant). If R stands for the measure of a rectangle from $x = x_0$ to $x = x_0 + a$ of unit height we can see that

$$\nu(\mathcal{R}_{n,m}) = \min\{\delta_1, \delta_2, |\lambda_j^{k_n} - \lambda_j^{k_m}|\} R$$

and since $\mathcal{R}_{n,m} \subset t_{\lambda_j^{k_n}}^1(A) * t_{\lambda_j^{k_m}}^1(A)$ we know that

$$(4) \quad \min\{\delta_1, \delta_2, |\lambda_j^{k_n} - \lambda_j^{k_m}|\} R \leq \nu(t_{\lambda_j^{k_n}}^1(A) * t_{\lambda_j^{k_m}}^1(A)).$$

The right side of Equation (4) is Cauchy with respect to m and n because $(t_{\lambda_j^{k_n}}^1)_{n=1}^\infty$ converges by Equation (3) and thus the left side is Cauchy as well. This means that $(\lambda_j^{k_n})_{n=1}^\infty$ is a Cauchy sequence of real numbers and thus must converge. Call its limit $\Lambda_j \in \mathbb{R}$. To complete the proof we must only show that $\nu(t_{\Lambda_j}^1(A) * A_j) = 0$. This is clear because

$$\nu(t_{\Lambda_j}^1(A) * A_j) \leq \nu(t_{\Lambda_j}^1(A) * t_{\lambda_j^{k_n}}^1(A)) + \nu(t_{\lambda_j^{k_n}}^1(A) * A_j)$$

and the right side goes to zero as $n \rightarrow \infty$. So we conclude that the original Cauchy sequence converges to $[(A, (\ell_{\Lambda_j}, +1, k_j)_{j=1}^{m_f})]$. Clearly the elements of each copy of $\mathbb{R}[[X, Y]]_0$ and $[0, 1]$

can be made to converge. The only problem is that possibly this limit does not have the critical points labeled in the correct order according to Remark 3.8 to be an element of $\widetilde{\mathcal{DPolyg}}_{m_f, [\vec{k}]}(\mathbb{R}^2)$ so we reorder it by some permutation $p \in \mathcal{S}^{m_f}$ and the result follows. \square

3.3. $\widetilde{\mathcal{M}}$ is complete.

Lemma 3.14. $\widetilde{\mathcal{M}}_{m_f, [\vec{k}]}$ is complete.

Proof. Any Cauchy sequence in $\widetilde{\mathcal{M}}_{m_f, [\vec{k}]}$ must have a subsequence which is Cauchy with respect to d^{Id} by Lemma 3.12. By Lemma 3.13 that sequence must converge with respect to d^p for some fixed $p \in \mathcal{S}^{m_f}$, which in particular means that it must converge with respect to d . A Cauchy sequence with a subsequence which converges must converge. \square

Now Lemma 3.11 and Lemma 3.14 imply the main result of this section.

Proposition 3.15. Given an admissible measure ν and a linear summable sequence the completion of $(\mathcal{M}, d^{\nu, \{b_n\}_{n=0}^\infty})$ is $(\widetilde{\mathcal{M}}, d^{\nu, \{b_n\}_{n=0}^\infty})$.

Remark 3.16. The reason to use d instead of d^{Id} can be seen by the examining structure of the completion. Let

$$[\Delta_w^l] = \begin{cases} [(\Delta_l, (\lambda_1 = 0, \epsilon_1 = 1, k_1 = 0), (\lambda_2 = l, \epsilon_2 = 1, k_2 = 0))] & \text{if } l > 0 \\ [(\Delta_l, (\lambda_1 = l, \epsilon_1 = 1, k_1 = 0), (\lambda_2 = 0, \epsilon_2 = 1, k_2 = 0))] & \text{if } l < 0 \end{cases}$$

and suppose that $m_l \in \mathcal{M}$ is a system given by

$$m_l = \begin{cases} ([\Delta_w^l], ((S_1)^\infty, h_1), ((S_2)^\infty, h_2)) & \text{if } l > 0 \\ ([\Delta_w^l], ((S_2)^\infty, h_2), ((S_1)^\infty, h_1)) & \text{if } l < 0 \end{cases}$$

for $l \in [-1, 1] \setminus \{0\}$ such that $\lim_{l \rightarrow 0} m_l$ exists in $(\widetilde{\mathcal{M}}, d)$. This can be thought of as one of the critical points being fixed and the other passing over it at $l = 0$ as is shown in Figure 11. The complications in defining this come from the fact that the order of the critical points switches at $l = 0$ so the labeling has to switch. Now we can see the problem with using d^{Id} , which is that $\lim_{l \rightarrow 0^+} m_l \neq \lim_{l \rightarrow 0^-} m_l$ with respect to d^{Id} .

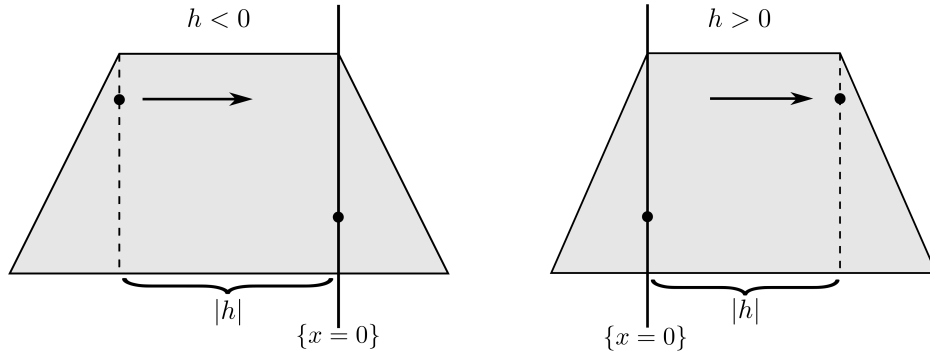


FIGURE 11. A continuous family in $\widetilde{\mathcal{M}}$ in which one critical point passes over the other as h increases from negative to positive.

Finally, Theorem A is produced by combining Proposition 2.13, Corollary 2.9, and Proposition 3.15.

4. FURTHER QUESTIONS

Now that we have defined a metric, and in particular a topology, on \mathcal{T} there are several questions that would be natural to address. First of all, one may be interested extending the metric defined in this paper in the way that this paper has extended the metric from [27]. To produce such an extension to a larger class of integrable systems one would first have to classify those systems with invariants in a way which extends the classification from [22, 23]. Now that we have defined a topology on \mathcal{T} would be natural to consider Problem 2.45 from [25], which asks what the closure of the set of semitoric integrable systems would be when considered as a subset of $C^\infty(M, \mathbb{R}^2)$. To address this problem an appropriate topology on $C^\infty(M, \mathbb{R}^2)$ would have to be defined. This situation is much more general than the systems which are the focus of this paper so it may be best to study metrics constructed in a more general case such as in [20].

It would of interest to try to better understand the elements of $\tilde{\mathcal{M}} \setminus \mathcal{M}$ in relation to integrable systems. Perhaps some subset of this can be interpreted as corresponding to non-simple semitoric systems or to some other type of integrable system not included in the classification by Pelayo and Vũ Ngọc in [22, 23]. Problem 2.44 from [25] asks if some integrable systems may be expressed as the limit of semitoric systems in an appropriate topology and the study of $\tilde{\mathcal{M}} \setminus \mathcal{M}$ may make some progress on this question.

Furthermore, now that a topology has been defined on \mathcal{T} questions regarding the continuity of functions on \mathcal{T} may be asked. For example, let $\Omega : \mathcal{T} \rightarrow [0, 1]$ be the *maximal semitoric ball packing density function* which assigns to each semitoric system the portion of the total volume of the manifold which may be approximated by disjoint semitoric symplectic balls. Before this question is explored there must be a definition of what it means for a ball to be symplectically embedded in a way which respects the structure of the semitoric system (similar to the definition of a toric ball packing in [9]).

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